Spiritual heights, which took us years of painful effort to attain, were attained by Zorba in one bound. And we said: "Zorba is a great soul!" Or else he leapt beyond those heights, and then we said: "Zorba is mad!"

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ON SOME PROPERTIES OF COMPOSITIONS OF AN INTEGER AND THEIR APPLICATION TO PROBABILITY THEORY AND STATISTICS

A THESIS

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ABSTRACT

The purpose of this thesis is to investigate combinatorial properties of partial orders defined on compositions of an integer, and to apply these properties to yield a unified approach to various problems in probability and statistics. The type of problems we deal with in probability are described as "ballot problems" by Feller. We show that ballot problems can conveniently be treated by the study of partial orders defined on compositions of an integer. As an application of the same ideas to problems in statistics, we consider several equivalent characterizations of simple sampling plans of size n and show that structurally, the problem of characterizing simple sampling plans is formally identical to the study of partial orders on compositions of an integer.

In Chapter I, a résumé of the relevant work on the compositions of an integer is presented. We extend and continue this work, bringing out more clearly its relation to ballot problems. Two simple and new methods are discussed, enabling us to rederive and extend most of the known results in this field.

In Chapter II, we prove that simple sampling plans of size n can be characterized by their boundary points. By the consistent use of the partial orders mentioned above, we obtain a 1:1 correspondence between simple sampling plans of size n and simple symmetric sampling plans of size $2n$. Certain topics and theorems which have been touched upon but not fully discussed in the literature are brought out as a natural consequence of our methods.

In Chapter III, application of the same concept has been made to other problems in probability theory and statistics. A brief mention of number-theoretic results which follow as a by-product of our approach concludes the chapter.

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CHAPTER 1

PARTIAL ORDERS AND BALLOT PROBLEMS

1.1 Introduction

The André-Poincaré "probleme du scrutin" [15] can be stated as follows: If m votes for A and n votes for B are counted one by one, $m \geq n$, what is the probability that A always leads? The algebraic solution $\frac{m-n}{m+n}$ of the problem has been obtained and interpreted from different approaches (Grossman [6], Feller [4], Narayana [10]). A natural extension of it that A holds an L-lead over B has also been solved (Narayana [11]). Dvoretzky and Motzkin [3] considered a remarkable generalisation of the ballot problem in another direction and this has subsequently been discussed by Grossman [7]. The problem is: If candidate A scores m votes and candidate B scores n votes, where $m > r\,n$, r being a non-negative integer, what is the probability that at each instant A's vote exceeds r times B's vote? We rederive the solution $\frac{m-rn}{m+n}$ by two different methods in sections 1.3 and 1.4.

1.2 Definitions and Notations

In this section, we give a few definitions, notations and a summary of relevant results [13] on partial orders defined on compositions. The idea of partial orders through the relation of domination defined below was initiated by Narayana in [12].

Composition of an integer:

Given an integer N, an r-composition of N $(t_1, t_2, \ldots, t_r)$ is a set of t_i where $t_i \geq 1$ is an integer for $i = 1, 2, \ldots, r$ such that

$$t_1 + t_2 + \ldots + t_r = N .$$

If r is an integer such that $1 \leq r \leq N$, there are obviously $\binom{N-1}{r-1}$ distinct r-compositions of N.

Relation of domination:

We say that an r-composition $(t_1, t_2, \ldots, t_r)$ of N dominates another r-composition $(t'_1, t'_2, \ldots, t'_r)$ of N, if and only if

$$\begin{aligned} t_1 &\geq t'_1 , \\ t_1 + t_2 &\geq t'_1 + t'_2 , \\ &\vdots \\ t_1 + \ldots + t_{r-1} &\geq t'_1 + \ldots + t'_{r-1} . \end{aligned} \qquad (A)$$

Evidently $t_1 + \ldots + t_r = t'_1 + \ldots + t'_r = N$.

Since the relation of domination defined by (A) is reflexive, transitive and antisymmetric, it represents a partial ordering of the r-compositions of N.

Composition vector:

Let $(T_1, T_2, \ldots, T_r)$ be the composition vector of r-elements associated with $(t_1, t_2, \ldots, t_r)$ such that

$$T_1 = t_1 ,$$

$$T_2 = t_1 + t_2 ,$$

$$\vdots$$

$$T_{r-1} = t_1 + \dots + t_{r-1} ,$$

$$T_r = t_1 + \dots + t_r = N .$$

The T's are integers satisfying

$$0 < T_1 < T_2 < \dots < T_r = N .$$

If $(t_1, t_2, \dots, t_r)$ dominates $(t'_1, t'_2, \dots, t'_r)$, we shall find it convenient to say that the composition vector $(T_1, T_2, \dots, T_r)$ of the former dominates the composition vector $(T'_1, T'_2, \dots, T'_r)$ of the latter. Since there is a 1:1 correspondence between the compositions and the corresponding composition vectors, we may consider only the vectors in what follows.

An important result concerning the above partial order is quoted here:

The number of r-compositions of N which are dominated by a particular r-composition, whose composition vector is $(T_1, T_2, \dots, T_r)$, is given by D_{r-1} in the following recursive formula:

$$D_k = \binom{T_k}{1} D_{k-1} - \binom{T_{k-1}+1}{2} D_{k-2} + \binom{T_{k-2}+2}{3} D_{k-3} - \dots + (-1)^{k-1} \binom{T_1+k-1}{k} D_0 , \tag{1}$$

where $D_0 = 1$.

This has been given in [13] without proof and the proof is indicated below.

Proof: When $r = 1$, we notice that N dominates itself and therefore $D_0 = 1$. Let $N(S)$ denote the number of elements in a set S. Further, let S_k be the set of vectors of k positive integers $(n_1, n_2, \ldots, n_k)$ defined as follows:

$$S_k = (n_1, n_2, \ldots, n_k : n_1 < n_2 < \ldots < n_k,\ n_1 \leq T_1,\ n_2 \leq T_2, \ldots, n_k \leq T_k)$$

where $0 < T_1 < T_2 < \ldots < T_k$, $k = 1, 2, \ldots, r-1$.

Obviously $D_k = N(S_k)$, $k = 1, 2, \ldots, r-1$. Then by the principle of inclusion and exclusion,

$$\begin{aligned}
D_k = N(S_k) &= N(n_k : n_k \leq T_k)\, N(n_1, n_2, \ldots, n_{k-1} : n_1 < n_2 < \ldots < n_{k-1}, \\
&\qquad n_1 \leq T_1,\ n_2 \leq T_2, \ldots, n_{k-1} \leq T_{k-1}) \\
&- N(n_{k-1}, n_k : n_k \leq n_{k-1} \leq T_{k-1})\, N(n_1, n_2, \ldots, n_{k-2} : n_1 < n_2 < \ldots < n_{k-2}, \\
&\qquad n_1 \leq T_1,\ n_2 \leq T_2, \ldots, n_{k-2} \leq T_{k-2}) \\
&+ N(n_{k-2}, n_{k-1}, n_k : n_k \leq n_{k-1} \leq n_{k-2} \leq T_{k-2})\, N(n_1, n_2, \ldots, n_{k-3} : \\
&\qquad n_1 < n_2 < \ldots < n_{k-3},\ n_1 \leq T_1,\ n_2 \leq T_2, \ldots, n_{k-3} \leq T_{k-3}) \\
&- \ldots\ldots\ldots\ldots + (-1)^{k-1} N(n_1, n_2, \ldots, n_k : n_k \leq n_{k-1} \leq \ldots \leq n_1 \leq T_1)
\end{aligned}$$

$$= T_k\, N(S_{k-1}) - N(S_{k-2}) \sum_{n_{k-1}=1}^{T_{k-1}} \sum_{n_k=1}^{n_{k-1}} 1 + N(S_{k-3}) \sum_{n_{k-2}=1}^{T_{k-2}} \sum_{n_{k-1}=1}^{n_{k-2}} \sum_{n_k=1}^{n_{k-1}} 1$$

$$- \cdots\cdots\cdots + (-1)^{k-1} \sum_{n_1=1}^{T_1} \sum_{n_2=1}^{n_1} \cdots \sum_{n_{k-1}=1}^{n_{k-2}} \sum_{n_k=1}^{n_{k-1}} 1$$

$$= \binom{T_k}{1} D_{k-1} - \binom{T_{k-1}+1}{2} D_{k-2} + \binom{T_{k-2}+2}{3} D_{k-3} - \cdots + (-1)^{k-1} \binom{T_1+k-1}{k} D_0 .$$

This proves the result.

A theorem on compositions which has been proved in [13] is stated: The r-compositions of an integer N form a distributive lattice $(1 \leq r \leq N)$. Evidently this theorem implies: The r-compositions of N dominated by a particular r-composition of N also form a distributive lattice.

Since there is an obvious 1:1 correspondence between the r-compositions $(t_1, t_2, \ldots, t_r)$ of N, where we now allow for t_i, the value zero, and the r-compositions $(t_1+1, t_2+1, \ldots, t_r+1)$ of $N+r$, the results for r-compositions of n where $t_i \geq 0$ can be easily derived.

1.3 Relation of Domination and its Connection with Ballot Problem

Using the partial order defined in 1.2, we solve the ballot problem discussed in 1.1. With this end in view we prove the following Theorem.

Theorem 1 : The number of $(p+1)$-compositions of N dominated by the $(p+1)$-composition $(a, \underbrace{b, b, \ldots, b}_{p-1}, N-a-(p-1)\,b)$ of

N is

$$(2) \qquad D_p(a,b) = \frac{a}{a+p(b-1)} \binom{a-1+pb}{p} = \frac{a}{a+pb} \binom{a+pb}{p}$$

for a , b and $N-a-(p-1)b > 0$.

<u>Proof:</u> The result is trivially true for $p = 0$, being equal to 1. The number of 2-compositions of N dominated by the 2-composition $(a, N-a)$ of n is a. Thus, the theorem is true for $p = 1$. We assume that it holds good for values up to $p-1$. Applying (1), we obtain

$$D_p(a,b) = \binom{a+(p-1)b}{1} \frac{a}{a+(p-1)(b-1)} \binom{a-1+(p-1)b}{p-1}$$

$$- \binom{a+(p-2)b+1}{2} \frac{a}{a+(p-2)(b-1)} \binom{a-1+(p-2)b}{p-2} + \dots + (-1)^{p-1} \binom{a+p-1}{p}$$

$$= \sum_{s=1}^{p} (-1)^{s-1} \binom{a+(p-s)b+s-1}{s} \frac{a}{a+(p-s)(b-1)} \binom{a-1+(p-s)b}{p-s}$$

$$= \frac{a}{p} \sum_{s=1}^{p} (-1)^{s-1} \binom{a+(p-s)b+s-1}{p-1} \binom{p}{s}$$

$$(3) \quad = -\frac{a}{p} \sum_{s=0}^{p} (-1)^{s} \binom{a+(p-s)b+s-1}{p-1} \binom{p}{s} + \frac{a}{a+pb} \binom{a+pb}{p} \quad .$$

Now consider the following expansion:

$$(1-x^{b-1})^p \ (1-x)^{-p} = \sum_{s=0}^{p} (-1)^s \binom{p}{s} x^{s(b-1)} \{ 1 + \binom{-p}{1}(-x) + \dots$$

$$+ \binom{-p}{a} (-x)^a + \ldots + \binom{-p}{a+b-1} (-x)^{a+b-1} + \ldots + \binom{-p}{a+2(b-1)} (-x)^{a+2(b-1)}$$

$$+ \ldots + \binom{-p}{a+p(b-1)} (-x)^{a+p(b-1)} + \ldots \} \ .$$

Collecting the coefficient of $x^{a+p(b-1)}$ in the right hand side of the above, we get

$$\sum_{s=0}^{p} (-1)^s \binom{p}{s} \binom{-p}{a+(p-s)(b-1)} (-1)^{a+(p-s)(b-1)}$$

$$= \sum_{s=0}^{p} (-1)^s \binom{a+(p-s)b+s-1}{p-1} \binom{p}{s} \ .$$

But the coefficient of $x^{a+p(b-1)}$ in $(1-x^{b-1})^p (1-x)^{-p}$ is zero. Hence upon substitution in the expression (3), we finally get the desired result

$$D_p(a,b) = \frac{a}{a+pb} \binom{a+pb}{p} = \frac{a}{a+p(b-1)} \binom{a-1+pb}{p} \ .$$

We will find it convenient to refer to this result as the <u>one</u> A.P. case, since in the composition vector $(T_1, T_2, \ldots, T_{p+1})$ corresponding to this case, the T's except T_{p+1} are in A.P. Generalisations of this result will be discussed later in this chapter. We give below two interpretations of the result in Theorem 1, (a) as the solution to an 'occupancy' problem, and (b) as the number of minimal lattice paths from the origin to a point, under certain restrictions. These interpretations are particularly useful in our discussion of the ballot problem and shed considerable light on the structure of simple sampling plans to be considered in Chapter II.

(a) Interpretation as an occupancy problem: Let $T(m,n; r)$ or briefly T denote the set of all $(n+1)$-compositions of N, where $N \geq m - r + n$ is a fixed integer, that are dominated by the $(n+1)$-composition $(m-rn, r+1, r+1, \ldots, r+1, N-m-n+r+1)$ of N. We note that T represents the set of composition vectors $(T_1, T_2, \ldots, T_n, N)$ dominated by the $(n+1)$-composition vector $(m-rn, m-r(n-1)+1, m-r(n-2)+2, \ldots, m-r+n-1, N)$. Let us define a new set $S(m,n; r)$, briefly S, of vectors $A = (a_1, a_2, \ldots, a_n)$ such that $a_i = T_i - i$. Obviously

$$0 \leq a_1 \leq a_2 \leq \ldots \leq a_n ,$$

and $$0 \leq a_i \leq (m-rn-1) + (i-1)r , \quad i = 1, 2, \ldots, n .$$

From the natural correspondence between elements of T and S, the relation of domination defined on T can be extended to S and since T is a distributive lattice [13], so is S.

Consider all possible distributions of indistinguishable objects in n ordered cells, satisfying the following conditions: If a_i $(i = 1, 2, \ldots, n)$ denotes the number of objects in the ith cell, the conditions are

$$0 \leq a_1 \leq a_2 \leq \ldots \leq a_n ,$$

and $$0 \leq a_i \leq (m-rn-1) + (i-1)r , \quad i = 1, 2, \ldots, n .$$

We notice that to each distribution of objects satisfying the above conditions corresponds a vector in S and conversely. Thus the number of possible distributions, by Theorem 1, is

$$(4) \qquad D_n(m-rn,\ r+1) = \frac{m-rn}{m+n}\binom{m+n}{n} .$$

(b) Interpretation as minimal lattice paths and solution to the ballot problem: Using a well-known representation of the number of possible voting records, let a vote for A be represented by a unit horizontal step and a vote for B by a unit vertical step. Then the set of possible voting records is the set of minimal lattice paths from the origin to the point (m,n) and the number of such paths is $\binom{m+n}{n}$. To each lattice path from $(0,0)$ to (m,n) lying entirely below the line $x = ry$ (i.e. not touching it except at the origin) corresponds a voting record in which A's vote exceeds r times the votes scored by B at each instant and conversely. Let $L(m,n; r)$ or briefly L denote the set of all lattice paths from $(0,0)$ to (m,n) which lie entirely below the line $x = ry$.

Let a , b be two paths $\in L$. We say that a dominates b (written $a\ D\ b$) if no part of b lies between a and the line $x = ry$. An illustration is provided for $r = 1$, $m = 7$, $n = 4$ in the figure 1 below.

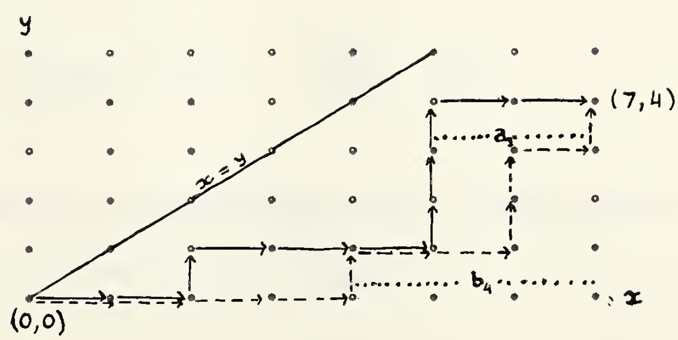
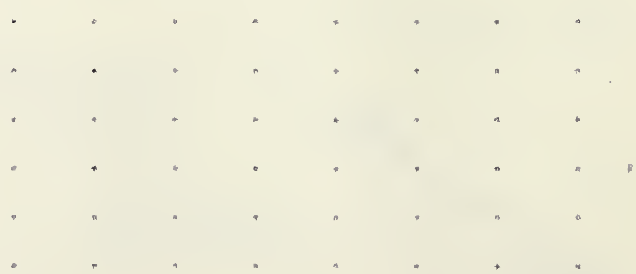


Figure 1

(Path a $\rightarrow$, path b $\dashrightarrow$, $a\ D\ b$)



As the relation D is reflexive, transitive and antisymmetric, it is a partial order defined on the elements of L.

We further observe that to each path $a \in L$ there corresponds a vector $A = (a_1, a_2, \ldots, a_n) \in S$ and conversely where the minimal distance measured parallel to the x-axis, of the point $(m,n-i)$ from the path a is a_i. For instance, the path a in the above figure corresponds to the vector $(2, 2, 2, 5)$ and conversely. Similarly the path b corresponds to the vector $(0, 1, 1, 3)$ and conversely. This 1:1 correspondence shows that the relation of domination defined on compositions and the relation D represent the same partial order and T , S and L are isomorphic partially ordered sets. From our earlier assertion, we state the following lemma:

<u>Lemma 1</u>: The sets $T(m,n; r)$, $S(m,n; r)$ and $L(m,n; r)$ of compositions, non-negative and non-decreasing vectors and lattice paths are isomorphic , being distributive lattices.

Because of this isomorphism between the lattices, the number of paths in L is

$$D_n(m-rn,\ r+1) = \frac{m-rn}{m+n} \binom{m+n}{n}$$

and therefore the probability that at each instant A's vote exceeds r times B's is $\frac{m-rn}{m+n}$.

1.4 An Alternative Approach to the Ballot Problem

We say that a p-composition $(t_1, t_2, \ldots, t_p)$ of m $[r]$-dominates a p-composition $(t'_1, t'_2, \ldots, t'_p)$ of n, if and only if

$$\begin{aligned} t_1 &\geq r\, t'_1 \\ t_1 + t_2 &\geq r\,(t'_1 + t'_2) \\ &\vdots \\ t_1 + \ldots + t_p &\geq r\,(t'_1 + \ldots + t'_p) \end{aligned} \qquad \text{(B)}$$

If $m < rn$, no p-composition of m can $[r]$-dominate a p-composition of n. If $m = rn$, the last inequality becomes an equality. Let us define $(m,n;\, r)_p$ as the number of lattice paths from $(0,0)$ to (m,n) not crossing the line $x = ry$, each path having exactly p horizontal and p vertical components. Obviously the horizontal components of each path define a p-composition of m and likewise the vertical components define a p-composition of n. Since the paths do not cross the line $x = ry$, to each path in $(m,n;\, r)_p$ corresponds a p-composition of m that $[r]$-dominates a p-composition of n and conversely. Then we state

<u>Theorem 2</u> :

$$(m,n;\, r)_p = \binom{m}{p-1}\binom{n-1}{p-1} - r\binom{m-1}{p-2}\binom{n}{p} \qquad (5)$$

<u>Proof:</u> Following [11] page 92, let $a_{i\alpha}$ be the increase in the ith coordinate $(i = 1, 2)$ at the αth step, such that,

(1) $a_{i\alpha} \geq 1$ for all i and α ,

(2) $$a_{11} \geq r\, a_{21} \;,$$

$$a_{11} + a_{12} \geq r\,(a_{21} + a_{22}) \;,$$

and in general for the αth step

$$\sum_{j=1}^{\alpha} a_{1j} \geq r \sum_{j=1}^{\alpha} a_{2j} \;.$$

For $p = 1$, obviously $(m,n;\, r)_1 = 1$ if $m \geq rn$, $n \geq 1$,

$= 0$ otherwise.

For $p = 2$, $(m,n;\, r)_2 = 0$ if $m < rn$ or $n < 2$,

$$(m,n;\, r)_2 = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} (i,j;\, r)_1 \,, \text{ if } m \geq rn,\ n \geq 2 \,,$$

$$= \sum_{i \geq rj}^{m-1} \sum_{j=1}^{n-1} 1 = m(n-1) - r\binom{n}{2} \;.$$

Thus for $p = 1, 2$, the theorem is true. So, applying induction, we get

$$(m,n;\, r)_p = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} (i,j;\, r)_{p-1} = \sum_{i \geq rj}^{m-1} \sum_{j=1}^{n-1} \left[\binom{i}{p-2}\binom{j-1}{p-2} - r\binom{i-1}{p-3}\binom{j}{p-1}\right]$$

$$= \binom{m}{p-1}\binom{n-1}{p-1} - \sum_{k=0}^{n-p} \binom{p+k-2}{p-2}\binom{r(p+k-1)}{p-1} - r\binom{m-1}{p-2}\binom{n}{p} + r\sum_{k=0}^{n-p} \binom{p+k-1}{p-1}\binom{r(p+k-1)-1}{p-2}$$

$$= \binom{m}{p-1}\binom{n-1}{p-1} - r\binom{m-1}{p-2}\binom{n}{p} \;,$$

since $$\sum_{k=0}^{n-p} \binom{p+k-2}{p-2}\binom{r(p+k-1)}{p-1} = \sum_{k=0}^{n-p} \binom{p+k-2}{p-2} \frac{r(p+k-1)}{p-1} \binom{r(p+k-1)-1}{p-2}$$

$$= r \sum_{k=0}^{n-p} \binom{p+k-1}{p-1} \binom{r\ (p+k-1)-1}{p-2} .$$

Hence the theorem is proved.

When $r = 1$,

$$(m,n;\ 1)_p = \binom{m}{p-1} \binom{n-1}{p-1} - \binom{m-1}{p-2} \binom{n}{p}$$

$$= \frac{(m-1)!}{(p-1)!\ (m-p+1)!} \quad \frac{(n-1)!}{p!\ (n-p)!} \quad [mp-n(p-1)]$$

$$= \frac{(m-1)!}{(p-1)!\ (m-p+1)!} \quad \frac{(n-1)!}{p!\ (n-p)\ !} \quad [p(m-p+1)-(p-1)(n-p)]$$

$$= \binom{m-1}{p-1} \binom{n-1}{p-1} - \binom{m-1}{p-2} \binom{n-1}{p} \quad ,$$

which coincides with the results in [11].

<u>Corollary</u>: The total number of minimal lattice paths from the origin to (m,n) , never crossing the line $x = ry$ is

$$\frac{m-rn+1}{m+n+1} \binom{m+n+1}{n} \quad .$$

<u>Proof</u>: Since any path to $(rn+\ell,\ n)$ $(\ell = 0,\ 1,\ \ldots,\ m-rn)$ in p steps $(p = 1,\ 2,\ \ldots,\ n)$ can be a path to (m,n) by joining the horizontal line from $(rn+\ell,\ n)$ to (m,n) , the required number is given by:

$$\sum_{p=1}^{n}\sum_{\ell=0}^{m-rn}(rn+\ell,\ n;\ r)_p = \sum_{p=1}^{n}\sum_{\ell=0}^{m-rn}\left[\binom{rn+\ell}{p-1}\binom{n-1}{p-1} - r\binom{rn+\ell-1}{p-2}\binom{n}{p}\right]$$

$$= \sum_{p=1}^{n}\left[\binom{n-1}{p-1}\left\{\binom{m+1}{p} - \binom{rn}{p}\right\} - r\binom{n}{p}\left\{\binom{m}{p-1} - \binom{rn-1}{p-1}\right\}\right]$$

$$= \sum_{p=1}^{n}\left[\binom{n-1}{p-1}\binom{m+1}{p} - r\binom{n}{p}\binom{m}{p-1}\right] = \binom{m+n}{m} - r\binom{m+n}{m+1} = \frac{m-rn+1}{m+n+1}\binom{m+n+1}{n} .$$

Evidently, the total number of minimal lattice paths from the origin to (m,n), never touching the line $x = ry$ (i.e. the number of paths in $L(m,n;\ r)$) is

$$\frac{m-rn}{m+n}\binom{m+n}{n} .$$

Thus we have provided another proof of the ballot problem.

1.5 Further Generalisation of the Ballot Problem

As a natural generalisation, we next consider the <u>two</u> A.P. case and obtain the following:

<u>Theorem 3</u> : The number of $(p+q+1)$-compositions of N dominated by the $(p+q+1)$-composition $(a,\ \underbrace{b,\ b,\ \ldots,\ b}_{p-1},\ c,\ \underbrace{d,\ d,\ \ldots,\ d}_{q-1},$ $N-a-(p-1)b-c-(q-1)d)$ of N is given by

$$(6) \qquad D_{p,q}(a,b;\ c,d) = \sum_{k=0}^{q}(-1)^k \frac{a}{a+(p+q-k)b}\binom{a+(p+q-k)b}{p+q-k}\frac{(q-k+1)b-c-(q-k)d}{(q-k+1)b-c-qd+k}\binom{(q-k+1)b-c-qd+k}{k}$$

for a, b, c, d, and $N-a-(p-1)b-c-(q-1)d > 1$.

When $q = 0$, $D_{p,q}(a,b;\ c,d)$ reduces to the one A.P. case i.e. $D_p(a,b)$.

<u>Proof</u>: We apply the recursive formula (1) to obtain the result. Let D_j have the same meaning as in section 1.2. Then by Theorem 1,

$$D_j = \frac{a}{a+jb}\binom{a+jb}{j} \quad \text{for} \quad j = 0, 1, 2, \ldots, p .$$

Now

$$D_{p+1} = \binom{a+(p-1)b+c}{1}\frac{a}{a+pb}\binom{a+pb}{p} - \binom{a+(p-1)b+1}{2}\frac{a}{a+(p-1)b}\binom{a+(p-1)b}{p}$$

$$+ \ldots + (-1)^p \binom{a+p}{p+1}$$

$$= \sum_{k=1}^{p+1} (-1)^{k-1}\binom{a+(p-k+1)+k-1}{k}\frac{a}{a+(p-k+1)b}\binom{a+(p-k+1)b}{p-k+1}$$

$$- (b-c)\frac{a}{a+pb}\binom{a+pb}{p}$$

$$= \frac{a}{a+(p+1)b}\binom{a+(p+1)b}{p+1} - (b-c)\frac{a}{a+pb}\binom{a+pb}{p}$$

the first term being obtained as in the proof of Theorem 1.

$$D_{p+2} = \binom{a+(p-1)b+c+d}{1}\left[\frac{a+1}{a+(p+1)b}\binom{a+(p+1)b}{p+1} - \frac{(b-c)a}{a+pb}\binom{a+pb}{p}\right]$$

$$- \binom{a+(p-1)b+c+1}{2}\frac{a}{a+pb}\binom{a+pb}{p} + \ldots + (-1)^{p+1}\binom{a+p+1}{p+2}$$

[illegible]

[illegible]

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$$[illegible]$$

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$$= \sum_{k=1}^{p+2} (-1)^{k-1} \binom{a+(p-k+2)b+k-1}{k} \frac{a}{a+(p-k+2)b} \binom{a+(p-k+2)b}{p-k+2}$$

$$- \frac{a(2b-c-d)}{a+(p+1)b} \binom{a+(p+1)b}{p+1} + \frac{a}{a+pb} \binom{a+pb}{p} [\binom{a+pb+1}{2} - \binom{a+(p-1)b+c+1}{2}$$

$$- (b-c)(a+(p-1)b+c+d)]$$

$$= \frac{a}{a+(p+2)b} \binom{a+(p+2)b}{p+2} - \frac{(2b-c-d)a}{a+(p+1)b} \binom{a+(p+1)b}{p+1}$$

$$+ \frac{(b-c)(b-c-2d+1)}{2} \frac{a}{a+pb} \binom{a+pb}{p} .$$

Thus the theorem is true for $j = 0, 1, 2, \ldots, p+2$. Assuming this to be true for values up to $p+q-1$, we get

$$D_{p,q}(a,b;c,d) = D_{p+q} = \binom{a+(p-1)b+c+(q-1)d}{1} \sum_{k=0}^{q-1} (-1)^k \frac{a}{a+(p+q-k-1)b}$$

$$\binom{a+(p+q-k-1)b}{p+q-k-1} \frac{(q-k)b-c-(q-k-1)d}{(q-k)b-c-(q-1)d+k} \binom{(q-k)b-c-(q-1)d+k}{k}$$

$$- \binom{a+(p-1)b+c+(q-2)d+1}{2} \sum_{k=0}^{q-2} (-1)^k \frac{a}{a+(p+q-k-2)b} \binom{a+(p+q-k-2)b}{p+q-k-2}$$

$$\frac{(q-k-1)b-c-(q-k-2)d}{(q-k-1)b-c-(q-2)d+k} \binom{(q-k-1)b-c-(q-2)d+k}{k}$$

$$+ \ldots + (-1)^{p+q-1} \binom{a+p+q-1}{p+q} .$$

After some simplification,

$$D_{p,q}(a,b;c,d) = \sum_{k=1}^{p+q} (-1)^{k-1} \binom{a+(p+q-1)b+k-1}{k} \frac{a}{a+(p+q-k)b} \binom{a+(p+q-k)b}{p+q-k}$$

$$+ \sum_{k=1}^{q} (-1)^k \frac{a}{a+(p+q-k)b} \binom{a+(p+q-k)b}{p+q-k}$$

$$\Big[\binom{a-1+(p+q-k)b+k}{k} - \sum_{s=1}^{k} \binom{a+(p-1)b+c+(q-s)d+s-1}{s}$$

$$\frac{(q-k+1)b-c-(q-k)d}{(q-k+1)b-c-(q-s)d+k-s} \binom{(q-k+1)b-c-(q-s)d+k-s}{k-s}\Big] .$$

The first term gives $\frac{a}{a+(p+q)b} \binom{a+(p+q)b}{p+q}$, by proceeding on the same line as in the proof of Theorem 1. Next, we can show that

$$\sum_{s=0}^{k} \binom{a+(p-1)b+c+(q-s)d+s-1}{s} \frac{(q-k+1)b-c-(q-k)d}{(q-k+1)b-c-(q-s)d+k-s} \binom{(q-k+1)b-c-(q-s)d+k-s}{k-s}$$

$$= \binom{a-1+(p+q-k)b+k}{k}$$

by Hagen/Rothe formula in [17].

$$\therefore \quad D_{p,q}(a,b;c,d) = \frac{a}{a+(p+q)b} \binom{a+(p+q)b}{p+q}$$

$$+ \sum_{k=1}^{q} (-1)^k \frac{a}{a+(p+q-k)b} \binom{a+(p+q-k)b}{p+q-k} \frac{(q-k+1)b-c-(q-k)d}{(q-k+1)b-c-qd + k}$$

$$\binom{(q-k+1)b-c-qd+k}{k}$$

$$= \sum_{k=0}^{q} (-1)^k \frac{a}{a+(p+q-k)b} \binom{a+(p+q-k)b}{p+q-k} \frac{(q-k+1)b-c-(q-k)d}{(q-k+1)b-c-qd+k} \binom{(q-k+1)b-c-qd+k}{k} .$$

Hence the theorem is proved.

We have observed that the set $S(m,n;\ r)$ of non-negative and non-decreasing vectors A in the one A.P. case has interpretations as solutions to an occupancy problem and to the ballot problem. The result of Theorem 3 is therefore restated as a result on non-negative and non-decreasing vectors as follows:

Let $A_{p,q}(a,b;\ c,d)$ be the set of vectors $(a_1, a_2, \ldots, a_{p+q})$ such that

(i) $0 \leq a_1 \leq a_2 \leq \ldots \leq a_{p+q}$,

(C) (ii) $0 \leq a_\alpha \leq a + (\alpha-1)b$ for $\alpha = 1, 2, \ldots, p$; $a,b \geq 0$,

(iii) $0 \leq a_\alpha \leq c + (\alpha-p-1)d$ for $\alpha = p+1, \ldots, p+q;$

$c \geq a + (p-1)b;$ $d \geq 0$.

Then the number of vectors in $A_{p,q}(a,b;\ c,d)$ is

$D_{p,q}(a+1,\ b+1;\ c-a-(p-1)b+1,\ d+1) = N_{p,q}(a,b;\ c,d)$ (say).

A few special cases of the above which are of interest and which are applied later to derive various results are discussed here.

(a) When the upper bound of a_α is in A.P. for the first p elements of the vectors and then becomes a constant for

the remaining elements i.e., when $d = 0$, we get $N_{p,q}(a,b;\ c,0)$

$$(7)\qquad = \sum_{k=0}^{q} (-1)^k \frac{a+1}{a+1+(p+q-k)(b+1)} \binom{a+1+(p+q-k)(b+1)}{p+q-k} \binom{a+(p+q-k)b-c}{k} .$$

(b) Let the elements in each vector be positive. Then the new set consisting of these positive and non-decreasing vectors is equivalent to

$$A_{p,q}(a,b;\ c,d) - A_{p-1,q}(a+b,\ b;\ c,d) .$$

The number of vectors in the set is therefore

$$N_{p,q}(a,b;\ c,d) - N_{p-1,q}(a+b,\ b;\ c,d) .$$

(c) Consider the set of vectors $(a_1,\ a_2,\ \ldots,\ a_{p+q})$ satisfying

(i) $a_\alpha = 0 \qquad \alpha = 1,\ 2,\ \ldots,\ p$;

(ii) $0 < a_{p+1} \le a_{p+2} \le \ldots \le a_{p+q}$;

(iii) $0 < a_\alpha \le c + (\alpha-p-1)d$ for $\alpha = p+1,\ \ldots,\ p+q;\ \ c > 0\ ,\ d \ge 0.$

This set can trivially be shown as equivalent to the set of vectors $(a_1,\ a_2,\ \ldots,\ a_q)$ such that

(i) $0 < a_1 \le a_2 \le \ldots \le a_q$;

(ii) $0 < a_\alpha \le c + (\alpha-1)d\ ,\ \ \alpha = 1,\ 2,\ \ldots\ q,\ c > 0\ ,\ d \ge 0$.

Further, either set is equivalent to

$$A_{p,q}(0,0;\ c,d) - A_{p+1,q-1}(0,0;\ c+d,d) .$$

Therefore the number of vectors in either is

$$(8)\qquad \sum_{k=0}^{q}(-1)^k \frac{c+(q-k)d}{c+qd}\binom{-c-qd}{k} - \sum_{k=0}^{q-1}(-1)^k \frac{c+d+(q-k-1)d}{c+d+(q-1)d}\binom{-c-d-(q-1)d}{k}$$

$$= (-1)^q \frac{c}{c+qd}\binom{-c-qd}{q} = \frac{c}{c+q(d+1)}\binom{c+q(d+1)}{q},$$

which is the same as $D_q(c,\ d+1)$.

A combinatorial identity which is of fundamental importance is considered here. Analogous to $A_{p,q}(a,b;\ c,d)$, we can define $A_p(a,b)$ for the one A.P. case as the set of vectors $(a_1, a_2, \dots, a_p)$ such that

$$0 \le a_1 \le a_2 \le \dots \le a_p \ ;$$

(D) and

$$0 \le a_\alpha \le a + (\alpha-1)b\ , \quad \alpha = 1, 2, \dots, p;\ a,b \ge 0\ .$$

The number of vectors in $A_p(a,b)$ can be obtained by Theorem 1, as

$$\frac{a+1}{a+1+p(b+1)}\binom{a+1+p(b+1)}{p}\ .$$

On the other hand, $A_p(a,b)$ consists of vectors

1) all of whose elements are positive,

2) all of whose elements except the first one are positive,

3) all of whose elements except the first two are positive,

. . . .

p) all of whose elements are zero.

Thus, by the application of (8) of the special case (c), we get the identity

$$\frac{a}{a+p(b+1)}\binom{a+p(b+1)}{p}+\frac{a+1}{a+1+(p-1)(b+1)}\binom{a+1+(p-1)(b+1)}{p-1}+\frac{a+2}{a+2+(p-2)(b+1)}$$

$$\binom{a+2+(p-2)(b+1)}{p-2}+\ldots+1=\frac{a+1}{a+1+p(b+1)}\binom{a+1+p(b+1)}{p},$$

or

$$\sum_{k=0}^{p}\frac{a+k}{a+k+(p-k)(b+1)}\binom{a+k+(p-k)(b+1)}{p-k}=\frac{a+1}{a+1+p(b+1)}\binom{a+1+p(b+1)}{p}. \tag{9}$$

The interpretation of $A_{p,q}(a,b;\ c,d)$ in terms of a certain set of minimal lattice paths is easily seen to be as follows: Let $L_{p,q}(a,b;\ c,d)$ be the set of minimal lattice paths from $(0,0)$ to $(c+qd+1,\ p+q)$ not touching (i) the line $x = dy$ between $(0,0)$ and (qd,q) beyond the origin and (ii) the line $x - (c+qd-a-pb) = b(y-q)$ beyond $(c+qd-a-pb,\ q)$. As an example, consider $p = 4$, $q = 3$, $a = 3$, $b = 2$, $c = 13$, $d = 1$.

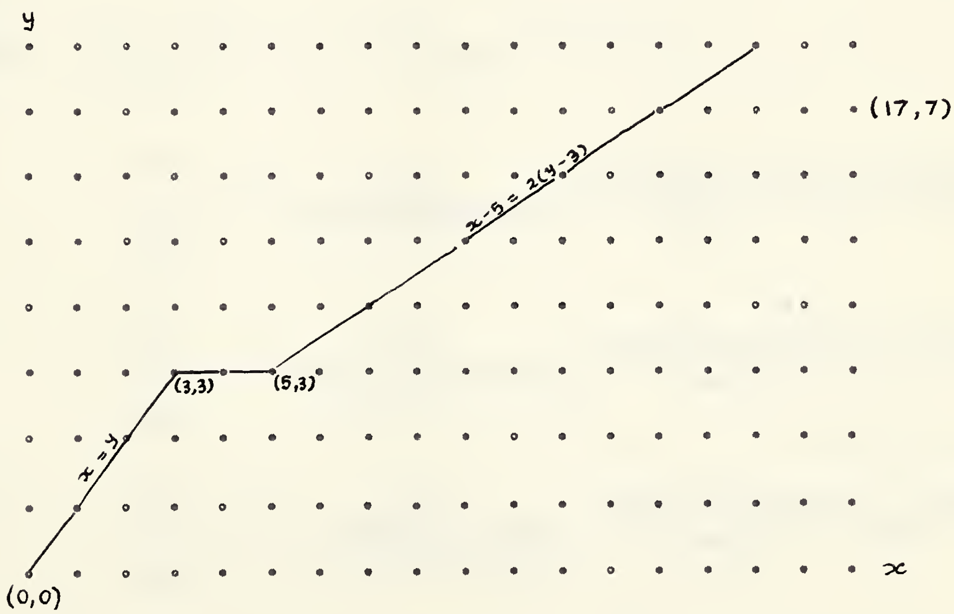

Figure 2

Any path from $(0,0)$ to $(17,7)$ below the marked boundary in Fig. 2 belongs to $L_{4,3}(3,2;\ 13,1)$. By a similar consideration as in the one A.P. case, we can show that there exists a 1:1 correspondence between the vectors in $A_{p,q}(a,b;\ c,d)$ and the paths $L_{p,q}(a,b;\ c,d)$ and that the two sets are isomorphic partial ordered sets.

Since any path to (x,y) is either through $(x-1,\ y)$ or $(x,\ y-1)$, it seems possible to get a recurrence relation for the two A.P. case, though we do not get a general one of a satisfactory form. However, we give as a lemma, a recurrence relation pertaining to the special case (a).

Lemma 2 :

(10) $$N_{p,q}(a,b;\ c-1,\ 0) + N_{p,q-1}(a,b;\ c,0) = N_{p,q}(a,b;\ c,0)$$

Proof: $N_{p,q}(a,b;\ c-1,\ 0) + N_{p,q-1}(a,b;\ c,0)$

$$= \sum_{k=0}^{q} (-1)^k \frac{a+1}{a+1+(p+q-k)(b+1)} \binom{a+1+(p+q-k)(b+1)}{p+q-k}\binom{a+(p+q-k)b-c+1}{k}$$

$$+ \sum_{k=0}^{q-1} (-1)^k \frac{a+1}{a+1+(p+q-k-1)(b+1)}\binom{a+1+(p+q-k-1)(b+1)}{p+q-k-1}\binom{a+(p+q-k-1)b-c}{k}$$

by (7),

$$= \sum_{k=1}^{q} (-1)^k \frac{a+1}{a+1+(p+q-k)(b+1)} \binom{a+1+(p+q-k)(b+1)}{p+q-k}\left[\binom{a+(p+q-k)b-c+1}{k} - \binom{a+(p+q-k)b-c}{k-1}\right] + \frac{a+1}{a+1+(p+q)(b+1)} \binom{a+1+(p+q)(b+1)}{p+q}$$

$$= \sum_{k=0}^{q} (-1)^k \frac{a+1}{a+1+(p+q-k)(b+1)} \binom{a+1+(p+q-k)(b+1)}{p+q-k} \binom{a+(p+q-k)b-c}{k}$$

$$= N_{p,q}(a,b;\ c,0) \qquad \text{by } (7).$$

Thus the lemma is proved.

We remark that the proof of the lemma can be established from the lattice path point of view.

Lastly we would like to mention that the results for the general r A.P. case can be obtained and the calculations are similar to thoseof the two A.P. case. We do not propose to present these because of the unwieldy nature of the expression.

CHAPTER II

CHARACTERISATION OF SIMPLE SAMPLING PLANS

2.1 Introduction

A formal description of binomial sampling plans was first given by Girshick, Mosteller and Savage in [5]. Minimum variance unbiased estimation for binomial sampling plans was discussed by DeGroot [2]. In a recent paper, Brainerd and Narayana [1] have proved the following result: The number of simple sampling plans of size n is $\frac{1}{n}\binom{3n}{n-1}$. The purpose of this chapter is to show the close connection between simple sampling plans and the partial orders described in Chapter I. We give several equivalent ways of characterising simple sampling plans which shed light on their structure and show that the structural aspects of the problems discussed in Chapters I and II are formally identical. These characterisations are useful in proving completeness of sampling plans when more than one parameter is involved.* Our approach also yields the result that the number of simple sampling plans of size n is $\frac{1}{n}\binom{3n}{n-1}$, without numerical calculations. It further clarifies and brings out ideas which have been touched upon but not fully discussed by earlier authors. To illustrate this point, we quote the following theorem: A necessary and sufficient condition for a sampling plan

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* I am thankful to Mr. I. Chorneyko, McMaster University, Hamilton, Canada for communicating results pertaining to completeness which he will publish shortly.

of size n to be simple is that it have exactly n+1 boundary points. An indirect proof of this exists in the literature, by the application of results in [2] and [5]. On the other hand, the theorem follows directly as a corollary from our method.

2.2 Some Auxiliary Results

For our purpose, we shall use the definitions and notations of [2] and make a distinction between essential and non-essential boundary points as follows: A boundary point (b.p.) γ_0 of a bounded sampling plan (s.p.) is essential if the s.p. S_{γ_0}, where $S_{\gamma_0}(\gamma) = S(\gamma)$ for $\gamma \neq \gamma_0$, and $S_{\gamma_0}(\gamma_0) = 1$, with γ_0 a continuation point (c.p.) is not bounded; A boundary point is non-essential otherwise. A boundary is essential if all its b.ps. are essential. It may be noted that the removal of any essential b.p. destroys closure in the sense of Lehman and Stein [8]. We also remark that if a point γ_0 is a non-essential b.p. of a s.p. S of size n, the boundary of S contains more than n+1 points, which follows from Theorem 8.1 of [2].

Let $C_k = \{(x,y) : x+y = k\}$, C be the set of all c.ps. and B the set of all b.ps. in a s.p. S. Further, let an accessible point be one which is not an inaccessible point (i.p.) i.e. which is either a b.p. or a c.p.. Some results which describe the properties of sampling plans of size n are stated and proved.

Lemma 3: Let S be a simple s.p. of size n. For each k and for each pair γ_1, γ_2 of b.ps. of S on C_k there are no inaccessible points (i.ps.) between γ_1 and γ_2.

Proof: Assume that γ_1 is above γ_2, i.e., γ_1 is farther from the x-axis than γ_2. Since $\gamma_i = (x_i, y_i) \in B$ $(i = 1, 2)$, γ_i can be reached by a path, implying that either $(x_i - 1, y_i)$ or $(x_i, y_i - 1)$ is a c.p. Thus $C_{k-1} \cap C$ is a non-void interval. The simplicity condition demands that if, for example, $\mu_1 = (x_1, y_1 - 1) \in C$ and $\mu_2 = (x_2 - 1, y_2) \in C$, all points of C_{k-1} between these two points μ_1, μ_2 are c.ps. If γ_3 (between γ_1, γ_2) is inaccessible, we have a contradiction, since all of the points in C_{k-1} between μ_1, μ_2 are c.ps.

Corollary: If either of the b.ps. γ_1, γ_2 in lemma 3 is a c.p., the lemma is still valid.

We remark that if S is a simple s.p. of size n, then S must contain at least two b.ps. of index n. If the s.p. contains two or more b.ps. of index n, these b.ps. must form an interval.

Lemma 4: Let $\gamma_0 = (x_0, y_0)$ be a b.p. of a s.p. S of size n. If $\gamma_1 = (x_0, y_0 + 1)$ and $\gamma_2 = (x_0 + 1, y_0)$ are accessible, then γ_0 is a non-essential b.p.

Proof: If γ_1, γ_2 are accessible, S_{γ_0} contains in addition to paths of S, those paths which pass through γ_0 and continue beyond it. Each of these additional paths contains either γ_1 or γ_2 and hence coincides with a path of S beyond either γ_1 or γ_2 as the

case may be. Thus the maximum length of paths in S_{γ_0} is bounded and thus γ_0 is non-essential.

<u>Lemma 5</u>: If γ_0 is a non-essential b.p. of a s.p. S of size n , then $C_{x_0 + y_0}$ contains at least one c.p.

<u>Proof</u>: If $C_{x_0 + y_0}$ does not contain any c.p., then all points in it are either b.ps. or i.ps. or both. So all points of index $\geq (x_0 + y_0)$ are inaccessible. This implies that γ_0 is essential which in turn proves the lemma.

<u>Theorem 4</u> : If S is bounded simple, then all its b.ps. are essential.

<u>Proof</u>: Let S be simple and contain a non-essential b.p. Let k be the smallest integer such that C_k contains a non-essential b.p. γ_0. Clearly $k > 1$. C_k contains at least one c.p. (lemma 5). Let $A = C_k \quad C$. Then A is an interval since S is simple. At one end of A , at least, lies either an essential b.p. or a non-essential b.p. (This follows from the Corollary to lemma 3.) Call this point $P = (x , y)$ and assume it is essential and a lower b.p., i.e., it is a b.p. which lies below the c.ps. on A. (See figure 3.)

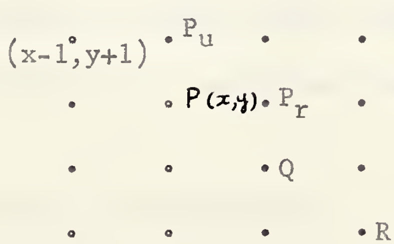

Figure 3

[illegible]

Lemma [illegible]

Proof: [illegible]

Lemma [illegible]

Proof: [illegible]

(see Figure [illegible]).

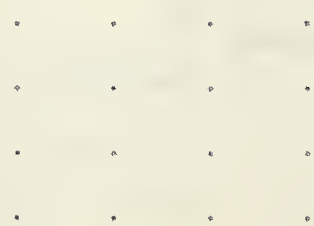

Figure [illegible]

[Note: $(x-1, y+1)$ is the last point on the lower end of A]

If the point $P_u = (x, y+1)$ is a c.p., then the point $P_r = (x+1, y)$ must be an i.p., and $Q=(x+1, y-1)$ is either a b.p. or an i.p. If Q is an i.p., then all points below Q in C_k are i.ps. (lemma 3). If Q is a b.p., then it is essential because P is essential. Consider next the point $R = (x+2, y-2)$ and repeat the argument until an i.p. is reached. The interval of b.ps. in this case is composed of essential b.ps.

The argument holds if P_u is a b.p. as well. (Clearly P_u is not an i.p. since $(x-1, y+1)$ is a c.p.) The modifications required when P is an essential upper b.p. are obvious.

Hence if P (which is contiguous to a c.p.) is an essential upper (lower) b.p., C_k cannot contain a non-essential upper (lower) b.p.

Assume the non-essential b.p. γ_0 to be a lower b.p. The above argument shows that γ_0 is contiguous with A. Then all points in C_k below γ_0 are either b.ps. or i.ps. Thus all points $(x_0 + k, y_0)$ $k = 1, 2, \ldots$ are i.ps. In this case S_{γ_0} would be unbounded and hence γ_0 would be essential.

A similar argument shows that γ_0 cannot be an upper b.p. (non-essential) also. Hence the theorem is proved.

2.3 The Concept of Deformation

We introduce the concept of deformation, which forms the basis of our characterisation of the simple sampling plans to be

dealt in the next section. Let S be a bounded s.p. with essential boundary and γ_0 be a b.p. of S. Consider a new s.p. S' as follows:

$$S'(\gamma) = S(\gamma) \quad \text{for all} \quad \gamma \neq \gamma_0 \ , \ (x_0+1, y_0) \ , \ (x_0, y_0+1)$$

and $S'(\gamma_0) = 1$. So far as (x_0+1, y_0) and (x_0, y_0+1) are concerned, only three cases can arise (lemma 4), viz.

(i) both are inaccessible;

(ii) one of these is inaccessible, the other a b.p.;

(iii) one of these is inaccessible, the other a c.p.

In case (i) either $S'(x_0+1, y_0) = 0$ or $S'(x_0, y_0+1) = 0$, but not both; in case (ii) if γ_1 is the i.p. and γ_2 the b.p., then $S'(\gamma_1) = S'(\gamma_2) = 0$; in case (iii) if γ_1 is the i.p. and γ_2 the c.p., then $S'(\gamma_1) = 0$, $S'(\gamma_2) = 1$. In other words, S' is constructed by shifting the b.p. γ_0 of S to one of the nearest i.ps. We then say that S' is a <u>deformation</u> of S at the b.p. γ_0. A deformation of S is possible at every boundary point.

S' is an <u>admissible</u> deformation of S if S' is bounded and its boundary is essential. We show the nontriviality of this definition, by providing examples of non-admissible deformation.

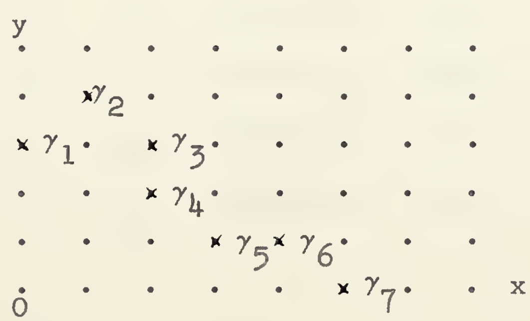

Figure 4

[illegible] positions, [illegible]

[illegible]

follows:

$f'(\gamma) = \gamma$, for all n [illegible] $+ \ldots +$ [illegible]

and $f'(\gamma) = 1$, [illegible] $+$; [illegible] $+$ [illegible]

[illegible], [illegible] the following [illegible].

(i) [illegible];

(ii) [illegible] 0 [illegible], [illegible];

(iii) [illegible] 0 [illegible], [illegible].

In case (i), since $f'(\gamma)$ [illegible] $=$ [illegible] $+$ [illegible] $=$ [illegible]; in case (ii) [illegible] γ_1 [illegible] γ [illegible], then $f'(\gamma_1) = f'(\gamma) =$ [illegible]; in case (iii), [illegible] γ_1 [illegible], [illegible] γ_1 [illegible], then $f'(\gamma_1) =$ [illegible], $f'(\gamma) = 1$. In [illegible], [illegible] [illegible] [illegible], γ_1 [illegible] [illegible] [illegible] points.

[illegible] [illegible] [illegible] [illegible]

of this definition, [illegible] [illegible] definition.

In Figure 4, S is defined by its b.ps. $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ and γ_7. The deformation of S at γ_3 is not bounded and the deformation at γ_4 has γ_5 as its non-essential b.p. These are therefore non-admissible deformations of S.

Given any simple s.p. of size n, it is possible to get the s.p. whose b.ps. have index n (i.e. single s.p. of size n), by a suitable choice of a series of admissible deformations. For simplicity, an illustration is provided below.

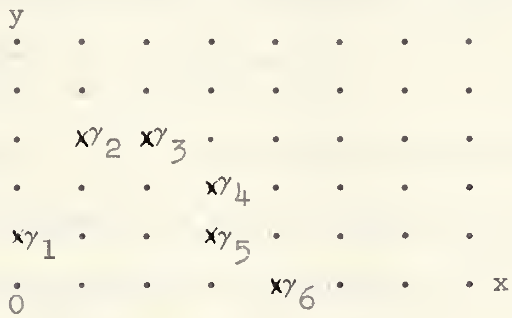

Figure 5

Let S be the simple s.p. whose b.ps. are $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ and γ_6 as indicated in Figure 5. We suggest the following series of admissible deformations:

S_1 : γ_1 shifted to (0,2) ;

S_2 : γ_1 shifted to (0,3) ;

S_3 : γ_1 shifted to (0,4) ;

S_4 : γ_2 shifted to (1,4) ;

S_5 : γ_1 shifted to (0,5) ;

S_6 : γ_5 shifted to (4,1) ;

S_7 : γ_6 shifted to (5,0) .

Evidently S_7 has the b.ps. of index n. In this process, γ_1 was pushed upwards 4 times, γ_2 once, γ_3 and γ_4 remained in their respective positions, γ_5 and γ_6 were pushed once each to the right. The number of times any b.p. is pushed to coincide with the corresponding b.p. of the single s.p. of the same size essentially defines the characterisation that we propose to discuss soon. For example, according to our characterisation the simple s.p. S corresponds to the vector (4, 1, 0, 0, 1, 1). Now we prove the existence of a sequence of admissible deformations for every simple s.p. of size n in Theorem 5.

Theorem 5 : If S is a simple s.p. of size n , then there exists a sequence of s.ps. $S = S_0, S_1, S_2, \ldots , S_k$ such that S_{i+1} is an admissible deformation of S_i $(i = 0, \ldots , k-1)$, S_k has exactly the points of index n as b.ps., and each S_i has n+1 b.ps.

Proof: Let B_0 be the boundary of S_0. Let k_0 be the smallest integer such that $C_{k_0} \cap B_0$ is not empty (assume $k_0 \neq n$). The c.ps. on C_{k_0} form an interval A_0 which is not empty. If there is a contiguous b.p. $\gamma_0 = (x_0, y_0)$ on C_{k_0} above A_0 , then (x_0+1, y_0) is an accessible point. Hence the deformation S_1 of S_0 at γ_0 has $S_1(\gamma) = S(\gamma)$ except for $\gamma \neq \gamma_0$, and $\gamma \neq (x_0, y_0+1)$. At these points γ_0 , (x_0, y_0+1) , S_1 differs from S_0. Above γ_0 on C_{k_0} , there are no i.ps. of S , because there are no b.ps. on lines C_k , $k < k_0$. Thus above γ_0 on C_{k_0} are only b.ps.

Since the only change in S_1 from S_0 is to shift the b.p. γ_0 to (x_0, y_0+1) making γ_0 a c.p., it is easily seen that S_1 is bounded if S_0 is bounded. Thus we have proved that S_1 is admissible.

In S_1, the interval of c.ps. on C_{k_0} contains one more point (viz. γ_0) than S_0 and the other c.ps. are those of S_0. A repetition of this process shifts the b.ps. one by one (whether from above or below A_0 on C_{k_0}) to the line C_{k_0+1}. If C_{k_0+1} has no c.ps., we have finished. Otherwise, continue the process. Induction proves that we finally reach the region S_k, where the b.ps. are exactly the points of index n. Clearly S_k and S_0 (in fact, every S_i) contain the same number of b.ps., namely $n+1$.

We next prove the converse of Theorem 5.

<u>Theorem 6</u> : If S is a non-simple s.p. of size n, then S contains more than $n+1$ b.ps.

<u>Proof</u>: Suppose S is of size n, non-simple and has non-essential b.ps. Thus from our earlier remark (at the beginning of 2.2) the theorem is proved. We restrict ourselves, therefore, to the case where S has only essential b.ps.

Let k_0 be the smallest integer such that C_{k_0} intersects the c.ps. of S in a configuration which is not an interval. Since the boundary of S is essential, the following configuration occurs in C_{k_0}: $\gamma_0 = (x_0, y_0)$ is a c.p.,

$\gamma_i = (x_0+i,\ y_0-i) = (x_i,\ y_i)$, $i = 1, \ldots, k$ are b.ps., and $\gamma_{k+1} = (x_0+k+1,\ y_0-k-1)$ is a c.p. where $k \geq 2$. Clearly $(x_1+1,\ y_1)$ is an i.p. and $(x_1,\ y_1+1)$ is accessible. A deformation S_1 at γ_1 is of the form: $S_1(\gamma) = S(\gamma)$ except for $\gamma \neq \gamma_1$, $\gamma \neq (x_1+1,\ y_1)$; at these points,

$$S_1(\gamma_1) = 1 \ ; \quad S_1(x_1+1,\ y_1) = 0 \ .$$

The deformation S_1 is bounded because possible candidates for paths of arbitrary length must pass through γ_1 and any path through γ_1 either coincides with an S-path from $(x_1,\ y_1+1)$ onwards or stops at $(x_1+1,\ y_1)$.

In the same manner S_i can be constructed from S_{i-1}, $i = 2, \ldots, k-1$. In S_{k-1}, γ_0, $\gamma_1, \ldots, \gamma_{k-1}$ are c.ps. γ_k is a b.p., and γ_{k+1} is a c.p. Clearly γ_k is a non-essential b.p. of S_{k-1} which is bounded. Therefore the plan S' where $S'(\gamma) = S_{k-1}(\gamma)$, $\gamma \neq \gamma_k$,

$$S'(\gamma_k) = 1$$

is a bounded plan and hence contains more than n b.ps. Thus S_{k-1} contains at least $n+2$ b.ps. and hence so does S_0.

<u>Theorem 7</u> : A necessary and sufficient condition for a s.p. of size n to be simple is that it have exactly $n+1$ b.ps.

<u>Proof:</u> The theorem follows from Theorems 5 and 6.

2.4 Representation of Simple Sampling Plans by Vectors

In conjunction with the ideas on deformation, we present two equivalent characterisations of simple s.ps. of size n in this section. For every positive integer n, we consider the set A_n of all $(n+1)$-vectors $\underline{a}_n$ of non-negative integers satisfying the following conditions:

(a) There exists an integer i, $1 \leq i \leq n$, such that $a_i = a_{i+1} = 0$ i.e. at least two consecutive a's in the vector $\underline{a}_n = (a_1, a_2, \ldots, a_{n+1})$ are zero.

(b) Let k be the smallest integer satisfying $a_i = a_{i+1} = 0 (1 \leq k \leq n)$. Then

$$a_1 \geq a_2 \geq \ldots \geq a_{k-1} > 0$$

and

$$0 \leq a_{k+2} \leq a_{k+3} \leq \ldots \leq a_{n+1} \quad .$$

We note that the inequalities in (b) may be trivially satisfied e.g. if $k = 1$ (so that $a_1 = a_2 = 0$), then a_{k-1} does not exist and the first set of inequalities in (b) is trivial.

(c) Let us denote by B and C the vectors $(b_1, b_2, \ldots, b_{k-1})$ and $(c_1, c_2, \ldots, c_{n-k})$ where $b_i = a_i$ $(i = 1, 2, \ldots k-1)$ and $c_\ell = a_{n+2-\ell}$ $(\ell = 1, 2, \ldots, n-k)$. The b's thus stand for a's with indices $\leq k-1$, and the c's for a's with indices $\geq k+2$. Then $b_j \leq n-j$ and $c_\ell \leq n-\ell$. Further, assuming

$$b_j = n-j-r \qquad (r = 0, 1, \ldots, n-j-1) ,$$

$$c_p \leq \begin{cases} n-p & \text{for} \quad p = 1, 2, \ldots, r \\ n-p-j & \text{for} \quad p > r ; \end{cases}$$

and similarly, assuming

$$c_\ell = n-\ell-r \qquad (r = 0, 1, \ldots, n-\ell) ,$$

$$b_p \leq \begin{cases} n-p & \text{for} \quad p = 1, 2, \ldots r \\ n-p-\ell & \text{for} \quad p > r . \end{cases}$$

A remark similar to (b) applies for (c) also; viz. if $k = 1$ or n , one of the vectors B or C is vacuous and the conditions simplify. Clearly $b_1 \geq b_2 \geq \ldots \geq b_{k-1} > 0$, and $c_1 \geq c_2 \geq \ldots \geq c_{n-k} \geq 0$.

We consider an example as it allows us to see the import of condition (c), which implies that if b's are assigned fixed values satisfying (a), (b) and (c), they provide bounds on all the c's. Take $n = 7$, $k = 3$, $b_1 = 6$, $b_2 = 3$ (note that $b_1 \leq 6$, $b_2 \leq 5$). As $b_1 = 6$, we obtain $c_1 \leq 5$, $c_2 \leq 4$, $c_3 \leq 3$, $c_4 \leq 2$, and as $b_2 = 3$, we obtain $c_1 \leq 6$, $c_2 \leq 5$, $c_3 \leq 2$, $c_4 \leq 1$. Thus if $b_1 = 6$, $b_2 = 3$, then $c_1 \leq 5$, $c_2 \leq 4$, $c_3 \leq 2$, $c_4 \leq 1$. Similarly if c's are assigned fixed values, they provide bounds on all b's.

Further let us consider the set A'_n of all (n+1)-vectors of non-negative integers $(a_1, a_2, \ldots, a_{n+1})$ satisfying conditions (a), (b), and the following condition (c').

(c') Defining the set of vectors as in (c), let $b_i \in B$ $(i = 1, 2, \ldots, k-1)$ and $c_j \in C$ $(j = 1, 2, \ldots, n-k)$ be arranged in non-decreasing order as $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$, keeping the convention: if some b's $\in B$ are equal to some c's $\in C$, the subscripts of α's corresponding to such b's $\in B$ are smaller than the subscripts of α's corresponding to c's $\in C$. Let

$$\beta_i = \begin{cases} 2\alpha_i - 1 & \text{if} \quad \alpha_i \in B \\ 2\alpha_i & \text{if} \quad \alpha_i \in C \end{cases} \qquad i = 1, 2, \ldots, n-1 .$$

Then

$$\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{n-1} ,$$

and

$$\beta_i \leq 2i \quad (i = 1, \ldots, n-1) .$$

Theorem 8 : (The 1:1 correspondence Theorem). The set A_n of vectors is identical with the set A'_n.

Before proving Theorem 8, we shall comment on its significance. The set A'_n of vectors satisfying (a), (b) and (c') is trivially in 1:1 correspondence with the set $A_{n-1}(2,2)$ defined in 1.5 (Chapter I). As we shall see in the next section, this Theorem provides an obvious 1:1 correspondence between simple s.ps. of size n and the simple symmetric s.ps. of size 2n.

Proof: Theorem 8 is proved by showing that any (n+1)-vector which satisfies (a), (b) and (c) also satisfies (a), (b) and (c') and conversely i.e. the conditions (c) and (c') are equivalent.

We first show that (c) implies (c'). Assuming $b_j = n-j-r$, it follows immediately from (b) that $k-j-1$ b's $\in B$ (viz. $b_{j+1}, b_{j+2}, \ldots, b_{k-1}$) precede b_j in the non-decreasing arrangement $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$. Again by (c), at least $n-k-r$ out of $n-k$ c's $\in C$ precede b_j in the same arrangement. Thus there are at least $n-j-r-1$ b's and c's which precede b_j. In other words, b_j can at most occupy one of the positions $n-j-r+s$ $(s = 0, 1, \ldots, j+r-1)$. Hence the corresponding β for b_j, which is equal to $2(n-j-r)-1$, is $\leq 2(n-j-r+s)$. This therefore implies (c'). Due to symmetry, a similar proof applies when $c_\ell = n-\ell-r$.

We next show that (c') implies (c). Since the discussion on c's $\in C$ is analogous to b's $\in B$, we prove only that (c') implies the first part of condition (c). From (c') $\beta_i \leq 2i \Leftrightarrow \alpha_i \leq i$. Further since $b_1 \geq b_2 \geq \ldots \geq b_{k-1}$, b_j can at most be α_{n-j}, which in turn $\leq n-j$. Thus every $b_j \leq n-j$. The only part left to be proved in (c) is: If $b_j = n-j-r$, then c_p for $p > r$ is bounded above by $n-p-j$. In this case, from (c'), b_j can be equal to one of the $\alpha_{n-j-r+t}$ $(t = 0, 1, \ldots, r)$. Clearly, the $j+r$ positions in the α-sequence from $n-j-r$ to $n-1$ are occupied by $b_1, b_2, \ldots, b_j$ and r c's. Noting that $c_1 \geq c_2 \geq \ldots \geq c_{n-k}$, these c's must be $c_1, c_2, \ldots, c_r$. Thus $c_{r+1}, c_{r+2}, \ldots, c_{n-k}$ can occupy positions in the α-sequence from 1 to $n-j-r-1$ only. Hence

$$c_{r+1} \leq n-j-r-1 \quad , \quad c_{r+2} \leq n-j-r-2 \quad ,$$

and so on. This implies that c_p for $p > r$ is bounded above by $n-p-j$.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

The vectors in A_n $(n = 1, 2, 3, 4)$ are given below.

n	A_n	n	A_n
1	(0, 0)	4 Cont'd.	(0, 0, 1, 1, 2)
			(0, 0, 1, 1, 3)
2	(0, 0, 0)		(0, 0, 1, 2, 2)
	(0, 0, 1)		(0, 0, 1, 2, 3)
	(1, 0, 0)		(1, 0, 0, 1, 1)
			(1, 0, 0, 1, 2)
3	(0, 0, 0, 0)		(1, 0, 0, 1, 3)
	(0, 0, 0, 1)		(1, 0, 0, 2, 2)
	(0, 0, 0, 2)		(1, 0, 0, 2, 3)
	(1, 0, 0, 0)		(2, 0, 0, 1, 1)
	(2, 0, 0, 0)		(2, 0, 0, 1, 2)
	(0, 0, 1, 1)		(2, 0, 0, 1, 3)
	(0, 0, 1, 2)		*(3, 0, 0, 1, 1)
	(1, 0, 0, 1)		*(3, 0, 0, 1, 2)
	(1, 0, 0, 2)		(1, 1, 0, 0, 1)
	(2, 0, 0, 1)		(2, 1, 0, 0, 1)
	(1, 1, 0, 0)		*(3, 1, 0, 0, 1)
	(2, 1, 0, 0)		*(2, 2, 0, 0, 1)
			*(3, 2, 0, 0, 1)
4	(0, 0, 0, 0, 0)		(1, 1, 0, 0, 2)
	(0, 0, 0, 0, 1)		(2, 1, 0, 0, 2)
	(0, 0, 0, 0, 2)		*(3, 1, 0, 0, 2)
	(0, 0, 0, 0, 3)		(1, 1, 0, 0, 3)
	(1, 0, 0, 0, 0)		(2, 1, 0, 0, 3)
	(2, 0, 0, 0, 0)		*(1, 1, 1, 0, 0)
	*(3, 0, 0, 0, 0)		*(2, 1, 1, 0, 0)
	(0, 0, 0, 1, 1)		*(3, 1, 1, 0, 0)
	(0, 0, 0, 1, 2)		*(2, 2, 1, 0, 0)
	(0, 0, 0, 1, 3)		*(3, 2, 1, 0, 0)
	(0, 0, 0, 2, 2)		
	(0, 0, 0, 2, 3)		
	(1, 0, 0, 0, 1)		
	(1, 0, 0, 0, 2)		
	(1, 0, 0, 0, 3)		
	(2, 0, 0, 0, 1)		
	(2, 0, 0, 0, 2)		
	(2, 0, 0, 0, 3)		
	*(3, 0, 0, 0, 1)		
	*(3, 0, 0, 0, 2)		
	(1, 1, 0, 0, 0)		
	(2, 1, 0, 0, 0)		
	*(3, 1, 0, 0, 0)		
	*(2, 2, 0, 0, 0)		
	*(3, 2, 0, 0, 0)		
	(0, 0, 1, 1, 1)		

(Note: The starred vectors will be discussed later.)

Let a simple s.p. of size n be represented by a vector whose elements are the distances of the b.ps. from the corresponding b.ps. of the single s.p. of size n (see discussion in 2.3). We may check easily that the set of vectors representing the s.ps. of size n $(n = 1, 2)$ is the same as the set A_n. In fact, we proceed to prove in the next Theorem that this is true for any n.

Theorem 9 : To every vector in A_n corresponds a simple s.p. of size n and conversely.

Proof: We have already verified that the Theorem is true for $n = 1$ and 2. Assume that it holds good for vectors $\underline{a}_{n-1} = (a_1, a_2, \ldots, a_n) \in A_{n-1}$. We shall prove inductively that it holds good for vectors $\underline{a}_n \in A_n$. From the geometrical representation, we already know that omission of a_1 or a_{n+1} in $\underline{a}_n$ corresponds respectively to a step to the right or a step upward from the origin. Thus for the proof, we are required to show that if either a_1 or a_{n+1} in $\underline{a}_n \in A_n$ is dropped out, the resulting vector should correspond to a simple s.p. of size $n-1$. For this purpose, we define a partition on A_n as A_n^o and A_n^*, where

$$A_n^o = \{ \underline{a}_n : a_j = n-j-r \ , \ j \leq k-1, \ r > 0 \} \ ,$$

and

$$A_n^* = \{ \underline{a}_n : a_j = n-j \ \text{ for at least one } \ j \leq k-1 \} \ .$$

(The starred vectors of A_4 in the above list belong to A_4^* and

the remaining to A_4^o). It can be easily shown that if a_{n+1} is omitted in any $\underline{a}_n \in A_n^o$ or a_1 (but not a_{n+1}) in any $\underline{a}_n \in A_n^*$, the resulting vector in either case satisfies conditions (a), (b) and (c), corresponding to vectors $\underline{a}_{n-1} \in A_{n-1}$ and therefore it belongs to A_{n-1}. Thus according to our hypothesis it corresponds to a simple s.p. of size n-1. (Vectors in A_4^o and A_4^* can be provided as examples to illustrate the above idea.)

From the concept of deformation, we have seen that to each simple s.p. of size n corresponds a vector of $n+1$ non-negative integers. Our discussion and the properties of simple s.ps. will further assert that this vector belongs to A_n. This completes the proof of the Theorem.

2.5 Simple Symmetric Sampling Plans and Concluding Remarks

We shall study a special class of simple s.p. called simple symmetric s.p. which is defined as follows:

A simple symmetric s.p. of size n is a simple s.p. of size n in which the b.ps. are symmetric about $y = x$.

We discuss these simple symmetric s.ps. of even and odd size separately. It immediately follows from the definition that in the vector $(a_1, a_2, \ldots, a_{2n+1})$ corresponding to any simple symmetric s.p. of size $2n$ $(n = 1, 2, \ldots)$,

$$(11) \qquad a_i = a_{2n+2-i} \quad (i = 1, 2, \ldots, n) \quad .$$

In conjunction with condition (11), conditions (a), (b) and (c) of the previous section in turn imply

$$a_n = a_{n+1} = a_{n+2} = 0 ,$$

$$\left.\begin{aligned} & a_i \geq a_{i+1} \quad (\text{or} \quad a_{2n+2-i-1} \leq a_{2n+2-i}) , \\ \text{and} \qquad & a_i (= a_{2n+2-i}) \leq 2n-2i , \end{aligned}\right\} \quad i = 1, 2, \ldots, n-1$$

respectively. Introducing for simplicity, the vector $(c_1, c_2, \ldots, c_{n-1})$ where $c_i = a_{n-i}$ to represent a simple symmetric s.p. in place of $(a_1, a_2, \ldots, a_{2n+1})$, we get the conditions on c's as:

$$\text{(d)} \qquad \text{and} \qquad \begin{aligned} & 0 \leq c_1 \leq c_2 \leq \ldots \leq c_{n-1} ; \\ & c_i \leq 2i , \quad i = 1, 2, \ldots, n-1. \end{aligned}$$

Similar consideration leads to the vectorial representation $(d_1, d_2, \ldots, d_n)$ of any simple symmetric s.p. of size $2n+1$ $(n = 0, 1, 2, \ldots)$ instead of $(a_1, a_2, \ldots, a_{2n+2})$ where $d_i = a_{n-i+1}$ (note $a_{n+1} = a_{n+2} = 0$), and the conditions on d's simplify to

$$\text{(e)} \qquad \text{and} \qquad \begin{aligned} & 0 \leq d_1 \leq d_2 \leq \ldots \leq d_n ; \\ & d_i \leq 2i-1 , \quad i = 1, 2, \ldots n . \end{aligned}$$

We prove the following results which follow as a natural consequence of our earlier discussion (including Chapter I).

<u>Theorem 10</u> : (i) The number of simple symmetric s.ps. of size $2n$ $(n = 1, 2, \ldots)$ and $2n+1$ $(n = 0, 1, 2, \ldots)$ are respectively $\frac{1}{n}\binom{3n}{n-1}$ and $\frac{1}{n+1}\binom{3n+1}{n}$;

(ii) The number of simple s.ps. of size n $(n = 1, 2, \ldots)$ is $\frac{1}{n}\binom{3n}{n-1}$.

Proof: Let S_n and S'_n denote the set of vectors $(c_1, c_2, \ldots, c_{n-1})$ and $(d_1, d_2, \ldots, d_n)$ respectively. Recalling the definition of the set $A_p(a,b)$ with the condition (D) in section 1.5, we note that S_n and S'_n are identical with $A_{n-1}(2,2)$ and $A_n(1,2)$ respectively. Therefore as a special case of Theorem 1, we obtain Theorem 10 (i).

On comparison of condition (d) above with (c') on the set A'_n in the previous section, it is clear from Theorems 8 and 9 that there exists a 1:1 correspondence between the simple symmetric s.ps. of size $2n$ and the simple s.ps. of size n. Hence the result of Theorem 10 (ii) follows.

We shall conclude this chapter with some remarks on the 1:1 correspondence Theorem, explicitly bringing out its close connection with the lattice-theoretic ideas considered earlier. We recall the definitions and some properties of the sets S_n and A_n - the vectors representing simple symmetric s.ps. of size $2n$ and simple s.ps. of size n - as we shall be mainly concerned with them in what follows:

S_n is the set of vectors $(c_1, c_2, \ldots, c_{n-1})$ satisfying

$$0 \leq c_1 \leq c_2 \leq \ldots \leq c_{n-1} ,$$

and

$$c_i \leq 2i , \qquad i = 1, 2, \ldots, n-1.$$

A_n is the set of vectors $(a_1, a_2, \ldots, a_{n+1})$ satisfying conditions (a), (b) and (c) in Section 2.4 of this chapter. The set S_n forms a distributive lattice under the order relation of domination viz : If $\underline{c}_n = (c_1, c_2, \ldots, c_{n-1})$ and $\underline{c}'_n = (c'_1, c'_2, \ldots, c'_{n-1})$ belong to S_n, we say $\underline{c}_n \, D \, \underline{c}'_n$ if $c_i \geq c'_i$ $(i = 1, 2, \ldots, n-1)$. An elegant interpretation of this is: If $\underline{c}_n \, D \, \underline{c}'_n$, the simple symmetric s.p. corresponding to $\underline{c}_n$ is contained in the simple symmetric s.p. corresponding to $\underline{c}'_n$. We notice that the symmetry of a simple symmetric s.p. allows us to get simultaneous deformation at two symmetrically placed b.ps. Keeping this in view, an alternate inductive proof similar to that in Theorem 9, by shifting the origin to $(1,1)$ can be supplied for the result: To each vector $\underline{c}_n$ corresponds a simple symmetric s.p. of size $2n$.

Let E_n be the subset of S_n which is defined as

$$E_n = \{\underline{e}_n\} = \{\underline{c}_n : c_i = 2\ell \text{ ; } \ell \text{ a non-negative integer } \leq i \text{ , } i = 1, 2, \ldots, n-1\} ,$$

E_n corresponds to the simple symmetric s.ps., whose vectors contain as elements even integers. For brevity, we may call E_n as the set of even vectors $\underline{e}_n$. For any vector $\underline{c}_n = (c_1, c_2, \ldots, c_n) \in S_n$, consider the set $E_n(\underline{c}_n)$ of all upper bounds of $\underline{c}_n$ which belong to E_n. Then there exists a least element in $E_n(\underline{c}_n)$. We may designate this as the closest dominating even vector, or briefly, closest even vector of $\underline{c}_n$. With these definitions and notations, the 1:1 correspondence theorem may be interpreted from the lattice

theoretic point of view as follows: (The general discussion and a numerical illustration are given side by side).

General Case	Illustration
Consider any vector $\underline{c}_n = (c_1, c_2, \ldots, c_{n-1}) \in S_n$. Let $\underline{e}_n = (e_1, e_2, \ldots, e_{n-1}) \in E_n$ be the closest even vector of $\underline{c}_n$.	Consider the vector $(1,4,4,5,8,10,13) \in S_8$. Here the even vector is $(2, 4, 4, 6, 8, 10, 14)$.
Take the differences $(e_1-c_1, e_2-c_2, \ldots, e_{n-1}-c_{n-1})$ where each difference is either 0 or 1.	The differences are $(1, 0, 0, 1, 0, 0, 1)$.
Take half of e_i's $(\frac{e_1}{2}, \frac{e_2}{2}, \ldots, \frac{e_{n-1}}{2})$.	Half of $(2,4,4,6,8,10,14)$ $= (1, 2, 2, 3, 4, 5, 7)$.
Partition these $\frac{e_i}{2}$'s into two vectors (i) whose corresponding $e_i - c_i$ is 1, and (ii) otherwise.	Two partitioned vectors are $(1, 3, 7)$ and $(2, 2, 4, 5)$.
Arrange vector (i) in non-increasing order and (ii) in non-decreasing order.	The vectors are $(7, 3, 1)$ and $(2, 2, 4, 5)$ after the arrangement.
Construct a new vector $\underline{a}_n$ by inserting two zeros between (i) and (ii) and this $\underline{a}_n$ belongs to A_n.	The new vector in A_8 is $(7, 3, 1, 0, 0, 2, 2, 4, 5)$.

Since S_n forms a distributive lattice, the 1:1 correspondence between S_n and A_n suggests that the set A_n also

forms a distributive lattice. The relation of domination for vectors $\underline{a}_n \in A_n$ is perhaps not a 'natural' one. However, the 1:1 correspondence between S_n and A_n plays an important role in Theorem 10: we obtain from it the number of simple s.ps. of size n, without any calculation, knowing only the number of simple symmetric s.ps. of size $2n$. We shall utilise the same mapping to yield other interesting results. In general, using a characteristic P, let $S(n; 0)$, $S(n; 1)$, ..., $S(n; k-1)$ be a partition of S_n into k classes. The above mapping will lead to a corresponding partition of A_n, say $A(n; 0)$, $A(n; 1)$, ..., $A(n; k-1)$. It may happen that this partition has an interesting characteristic P'. Then we say P and P' are equivalent. However, in many occasions, P and P' turn out to be characteristics of similar nature. In such cases, $P(= P')$ will be called an invariant characteristic under the 1:1 mapping. Investigation of the invariance and equivalence property of a few characteristics will be made in Chapter III.

CHAPTER III

APPLICATIONS

3.1 Introduction

This chapter consists of three sections in which we give an account of some problems related to our approach. The first section elaborates the discussion of invariant (equivalent) characteristics that has been introduced at the end of the last chapter. It further includes the derivation of a few combinatorial identities, which arise as a natural consequence of the properties of these characteristics. The second section illustrates an application of our ideas to a coin tossing problem. This section is perhaps not directly connected with our previous development. However, we include it as another example of a device which is useful in proving the completeness of a family of distributions.* The concluding section mentions briefly certain number-theoretic results which are suggested by our approach.

3.2 Invariant or Equivalent Characteristics of Simple Sampling Plans

(i) At the first instance, let us consider the partition $S(n;0)$, $S(n;1)$, ..., $S(n;n-1)$ of S_n, based on the characteristic

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* A paper entitled "Minimum variance unbiased estimation in coin tossing problems" by Narayana and Sathe to appear shortly in Sankhyā employs this device. In what follows [*] will refer to this paper.

P that any vector $\underline{c}(n;i) \in S(n;i)$ $(i = 0, 1, \ldots, n-1)$ contains exactly the first i elements as zeros. By the application of the procedure mentioned in section 2.5, we observe that the above partition of S_n maps into the partition $A(n;0), A(n;1), \ldots, A(n;n-1)$ of A_n, where $\underline{a}(n;i) \in A(n;i)$ contains exactly $i+2$ elements as zeros. The characteristic P' relative to the partition $\{A(n;i)\}$ determines the number of zeros in the vectors belonging to any $A(n;i)$. If the two essential zero's (i.e. $a_k = a_{k+1} = 0$) are dropped from all the vectors in A_n, then any vector in $A(n;i)$ will have the same number of zeros as that of any vector in $S(n;i)$. With this convention, we may say that the number of zeros is invariant under the 1:1 mapping.

The evaluation of the number $N(S(n;i))$ of vectors in $S(n;i)$ (or in $A(n;i)$) is very simple and in fact, it follows directly from (8) in 1.5. Setting $c = 2i + 2$, $d = 2$ and $q = n - 1 - i$,

$$(12) \qquad N(S(n;i)) = \frac{2i+2}{3n-i-1}\binom{3n-i-1}{n-i-1}$$

$$= \frac{i+1}{n}\binom{3n-i-2}{n-i-1} \qquad i = 0, 1, \ldots, n-1 .$$

It is interesting to note that

$$(13) \qquad N(S(n;0)) = \frac{1}{n}\binom{3n-2}{n-1}$$

i.e. the number of simple symmetric s.ps. of size $2n$, whose

vectors contain no zeros, is equal to the number of simple symmetric s.ps. of size $2n-1$.

In a similar way, we can partition S'_n, the set of vectors representing the simple symmetric s.ps. of size $2n+1$, into $S'(n;0)$, $S'(n;1)$, ..., $S'(n;n)$ where any vector $\underline{c}'_i(n;i) \in S'(n;i)$ $(i = 0, 1, \ldots, n)$ contains exactly the first i elements as zeros. Let $N(S'(n;i))$ be the number of vectors in $S'(n;i)$. Then by (8), we obtain

$$(14) \qquad N(S'(n;i)) = \frac{2i+1}{2n+1} \binom{3n-i}{n-i} \qquad i = 0, 1, 2, \ldots, n .$$

A particular case of (14) is

$$(15) \qquad N(S'(n;0)) = N(S'(n;1)) = \frac{1}{n} \binom{3n}{n-1} ,$$

which implies that the number of simple symmetric s.ps. of size $2n+1$, whose vectors contain either no zeros or exactly one zero, is the same as the number of simple symmetric s.ps. of size $2n$.

(ii) We next consider another partition $T(n;0)$, $T(n;1)$, ..., $T(n; 2n-2)$ of S_n, relative to a characteristic Q, where any vector $\underline{t}(n;i) \in T(n;i)$, $i = 0, 1, \ldots, 2n-2$ has its last element equal to i. Let $T(n;i) \subset S_n$ map into $B(n;i) \subset A_n$ and let the characteristic of the partition $\{B(n;i)\}$ be Q'. For reasons which will be obvious from what follows, we investigate the properties of Q', separately for odd and even i.

When i is odd, the last element of the closest even vector (defined in 2.5) of $\underline{t}(n;i)$ is $i+1$ and therefore, the first element of the corresponding vector $\underline{b}(n;i) \in B(n;i)$ is $\frac{i+1}{2}$. Furthermore, since the largest even element in $\underline{t}(n;i)$ is $\leq i-1$, the last element of $\underline{b}(n;i)$ must be $\leq \frac{i-1}{2}$. When i is even, clearly the last element of $\underline{b}(n;i)$ is $\frac{i}{2}$. Also the largest odd element in $\underline{t}(n;i)$ is $\leq i-1$. Therefore, the first element of $\underline{b}(n;i)$ is $\leq \frac{i}{2}$. In short, if i is odd, every vector $\underline{b}(n;i)$ has the first element equal to $\frac{i+1}{2}$ and the last element $\leq \frac{i-1}{2}$; and if i is even, every vector $\underline{b}(n;i)$ has the last element equal to $\frac{i}{2}$ and the first element $\leq \frac{i}{2}$. This completes the description of the properties of Q'. Thus Q and Q' are not characteristics of similar nature and therefore we merely say that Q is equivalent to Q'.

Let $N(T(n;i))$ be the number of vectors in $T(n;i)$. Let $(n;r,c)$ denote the number of vectors in A_n, whose a_1 (the first element) $= r$, and a_{n+1} (the last element) $= c$, $r = 0, 1, \ldots, n-1$ and $c = 0, 1, 2, \ldots, n-1$. Then

$$(16) \qquad N(T(n;\, 2r-1)) = \sum_{j=0}^{r-1} (n;\, r,j), \quad r = 1, 2, \ldots, n-1\,;$$

and

$$(17) \qquad N(T(n;\, 2r)) = \sum_{j=0}^{r} (n;\, j,r), \quad r = 1, 2, \ldots, n-1.$$

Trivially, of course,

$$(18) \qquad N(T(n;0)) = (n;\, 0,0) = 1\,.$$

To evaluate $N(T(n;i))$, we may recall the two A.P. case (vide section 1.5) and note that $T(n;i)$ is equivalent to the set $A_{p,n-p-2}(2,2;\ i,0)$, where $p = \frac{i-1}{2}$ or $\frac{i}{2}$ according as i is odd or even. Using (7), we obtain

$$(19) \qquad N(T(n;i)) = \sum_{k=0}^{n-p-2} (-1)^k \frac{1}{n-k-1} \binom{3n-3k-3}{n-k-2} \binom{2n-2k-i-2}{k} .$$

Since $i = 2p$ or $2p+1$, $2n-2k-i-2 > 0$ for $0 \leq k \leq n-p-2$. Hence $\binom{2n-2k-i-2}{k} > 0$, only if $k \leq 2n-2k-i-2$ i.e. only if $k \leq [\frac{2n-i-2}{3}]$ (which is $\leq n-p-2$), where $[x]$ is the largest integer contained in x. Then (19) becomes,

$$(20) \qquad N(T(n;i)) = \sum_{k=0}^{[\frac{2n-i-2}{3}]} (-1)^k \binom{3n-3k-3}{n-k-2} \binom{2n-2k-i-2}{k} .$$

The expression (20) simplifies to

$$(21) \qquad N(T(n;i)) = \frac{1}{n-1} \binom{3n-3}{n-2} \quad \text{for } i = 2n-4,\ 2n-3,\ 2n-2 .$$

Thus the number of simple symmetric s.ps. of size $2n$, whose c_{n-1} (i.e. the last element in the vectorial representation) $= 2n-4,\ 2n-3,\ 2n-2$ is equal to the number of simple symmetric s.ps. of size $2n-2$. This fact can be directly verified, since all vectors in $T(n;i)$ $i = 2n-4,\ 2n-3,\ 2n-2$ can be obtained by inserting an element equal to i at the end of each vector in S_{n-1}.

Recalling Lemma 2 of 1.5, we establish the following recurrence relation:

$$(22) \qquad N(T(n;i)) = N(T(n;\ i-1) + N(T(n-1;i))$$

$$\text{for} \quad i = 0,\ 1,\ \ldots,\ 2n-2\ ; \quad n = 1,\ 2,\ \ldots$$

The boundary conditions are

$$(23) \qquad \begin{cases} N(T(n;i)) = 0 & \text{for } i \geq 2n-1\ , \quad \text{and all } n\ , \\ N(T(0;i)) = 0 & \text{for all } i\ , \\ \text{and} & \\ N(T(n;0)) = 1 & \text{for all } n\ . \end{cases}$$

The solution of (22) which provides another expression for $N(T(n;i))$ is given below:

$$(24) \qquad N(T(n;i)) = \binom{n+i-1}{i} - \sum_{k=1}^{[\frac{i+1}{2}]} \frac{1}{k-1} \binom{3k-3}{k-2} \binom{n+i-3k+1}{n-k}\ .$$

Therefore, we have the following identity:

$$(25) \qquad \sum_{k=0}^{[\frac{2n-i-2}{3}]} (-1)^k \frac{1}{n-k} \binom{3n-3k-3}{n-k-2} \binom{2n-2k-i-2}{k}$$

$$= \binom{n+i-1}{i} - \sum_{k=1}^{[\frac{i+1}{2}]} \frac{1}{k-1} \binom{3k-3}{k-2} \binom{n+i-3k+1}{n-k}\ .$$

Defining $N(T'(n;i))$ analogously for the set S'_n , we can obtain results similar to (20), (21) and (24). These, on the other hand, yield the identity

$$(26) \qquad \sum_{k=0}^{[\frac{2n-i-1}{3}]} (-1)^k \frac{1}{n-k} \binom{3n-2k-2}{n-k-1} \binom{2n-2k-i-1}{k}$$

$$= \binom{n+i-1}{i} - \sum_{k=1}^{[\frac{i}{2}]} \frac{1}{k} \binom{3k-2}{k-1} \binom{n+i-3k}{n-k} \quad .$$

The number of simple symmetric s.ps. of even and odd size having the last element in the vectorial representation as i are given in the following tables.

TABLE 1

Number of simple symmetric sampling plans of size $2n$ ($n = 1$ to 8) whose c_{n-1} (the last element in the vectorial representation; vide 2.5) = i.

n \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	3
3	1	2	3	3	3	0	0	0	0	0	0	0	0	0	0	12
4	1	3	6	9	12	12	12	0	0	0	0	0	0	0	0	55
5	1	4	10	19	31	43	55	55	55	0	0	0	0	0	0	273
6	1	5	15	34	65	108	163	218	273	273	273	0	0	0	0	1428
7	1	6	21	55	120	228	391	609	882	1155	1428	1428	1428	0	0	7752
8	1	7	28	83	203	431	822	1431	2313	3468	4896	6324	7752	7752	7752	43263

TABLE 2

Number of simple symmetric sampling plans of size $2n+1$ ($n = 0$ to 7) whose d_n (the last element in the vectorial representation; vide 2.5) = i.

n \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	2
2	1	2	2	2	0	0	0	0	0	0	0	0	0	0	7
3	1	3	5	7	7	7	0	0	0	0	0	0	0	0	30
4	1	4	9	16	23	30	30	30	0	0	0	0	0	0	143
5	1	5	14	30	53	83	113	143	143	143	0	0	0	0	728
6	1	6	20	50	103	186	299	442	585	728	728	728	0	0	3876
7	1	7	27	77	180	366	665	1107	1692	2420	3148	3876	3876	3876	21318

We mention some properties of $(n; r,c)$ defined above (see (16) and (17)), which yield further identities.

It directly follows from condition (c) of 2.5, that

(I) $(n; r,c) = 0$ for r or $c \geq n$,

(II) $(n; n-1, n-1) = 0$ for $n > 1$.

Condition (a) of 2.5 gives,

(III) $(1; 0,0) = 1$.

The symmetry of condition (c) on b_j and c_ℓ suggests that to every vector $\underline{a}_n \in A_n$ whose $a_1 = r$ and $a_{n+1} = c$ corresponds a vector $\underline{a}'_n \in A_n$ whose $a_1 = c$ and $a_{n+1} = r$. In fact, by reversing the order of the elements in the set $\{\underline{a}_n : a_1 = r, a_{n+1} = c\}$ of vectors, we get 1:1 correspondence between this set and the set $\{\underline{a}_n : a_1 = c, a_{n+1} = r\}$ of vectors. Geometrically, the s.p. corresponding to the vector $\underline{a}_n$ where $a_1 = r$ and $a_{n+1} = c$, is the reflection of the s.p. corresponding to the vector $\underline{a}'_n$ where $a'_1 = c$, $a'_{n+1} = r$, about the line $x = y$. Thus

(IV) $(n; r,c) = (n; c,r)$.

Every vector of the set $\{\underline{a}_n : a_1 = r, a_n = c \leq r\}$ $r = 0, 1, \ldots, n-2$ can be obtained without repetition or omission from every vector of the set $\bigcup_{k=0}^{r} \{\underline{a}_{n-1} : a_1 = k, a_{n-1} = c\}$, by inserting an element equal to r at the beginning of the vector. Therefore,

$$(V) \quad (n;\ r,c) = \sum_{k=0}^{r} (n-1;\ k,c) \qquad c \leq r\ , \quad r = 0,\ 1,\ \ldots,\ n-2\ .$$

A similar argument shows that

$$(VI) \quad (n;\ n-1,c) = (n;\ n-2,c) = \sum_{k=0}^{n-2} (n-1,\ k,c),\ c = 0,\ 1,\ \ldots,\ n-2.$$

(The case $c = n-1$ is covered by (II).)

The numbers $(n;\ r,c)$ can be arranged in a $n \times n$ matrix M_n, where the element in the ith row and jth column of M_n is $(n;\ i-1,\ j-1)$. M_n is symmetric by (IV). Starting with M_1, we can form recursively, $M_2, M_3, \ldots$, by using the properties of $(n;\ r,c)$. Table 3 provides such matrices for $n = 1, 2, \ldots, 8$.

TABLE 3

Table of M_n (n = 1 to 8)

Size	Matrix	Total number of plans
n = 1	$\begin{array}{cc} a_1 \backslash a_2 & \begin{array}{c} 0 \end{array} \\ \begin{array}{c} 0 \end{array} & \begin{pmatrix} 1 \end{pmatrix} \\ \text{Total} & \begin{array}{c} 1 \end{array} \end{array}$	1
n = 2	$\begin{array}{cc} a_1 \backslash a_3 & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ \text{Total} & \begin{array}{cc} 2 & 1 \end{array} \end{array}$	3
n = 3	$\begin{array}{cc} a_1 \backslash a_4 & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \\ \text{Total} & \begin{array}{ccc} 5 & 4 & 3 \end{array} \end{array}$	12
n = 4	$\begin{array}{cc} a_1 \backslash a_5 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 3 & 5 & 5 \\ 3 & 3 & 4 & 4 \\ 5 & 4 & 3 & 3 \\ 5 & 4 & 3 & 0 \end{pmatrix} \\ \text{Total} & \begin{array}{cccc} 14 & 14 & 15 & 12 \end{array} \end{array}$	55
n = 5	$\begin{array}{cc} a_1 \backslash a_6 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 4 & 9 & 14 & 14 \\ 4 & 9 & 10 & 14 & 14 \\ 9 & 10 & 12 & 15 & 15 \\ 14 & 14 & 15 & 12 & 12 \\ 14 & 14 & 15 & 12 & 0 \end{pmatrix} \\ \text{Total} & \begin{array}{ccccc} 42 & 48 & 61 & 67 & 55 \end{array} \end{array}$	273

Size	Matrix									Total number of plans
$n = 6$	$a_1 \backslash a_7$	0	1	2	3	4	5			
	0	1	5	14	28	42	42			
	1	5	10	20	34	48	48			
	2	14	20	31	46	61	61			
	3	28	34	46	55	67	67			
	4	42	48	61	67	55	55			
	5	42	48	61	67	55	0			
	Total	132	165	233	297	328	273			1428
$n = 7$	$a_1 \backslash a_8$	0	1	2	3	4	5	6		
	0	1	6	20	48	90	132	132		
	1	6	15	35	69	117	165	165		
	2	20	35	65	111	172	233	233		
	3	48	69	111	163	230	297	297		
	4	90	117	172	230	273	328	328		
	5	132	165	233	297	328	273	273		
	6	132	165	233	297	328	273	0		
	Total	429	572	869	1215	1538	1701	1428		7752
$n = 8$	$a_1 \backslash a_9$	0	1	2	3	4	5	6	7	
	0	1	7	27	75	165	297	429	429	
	1	7	21	56	125	242	407	572	572	
	2	27	56	120	231	403	636	869	869	
	3	75	125	231	391	621	918	1215	1215	
	4	165	242	403	621	882	1210	1538	1538	
	5	297	407	636	918	1210	1428	1701	1701	
	6	429	572	869	1215	1538	1701	1428	1428	
	7	429	572	869	1215	1538	1701	1428	0	
	Total	1430	2002	3211	4791	6599	8298	9180	7752	43263

We find an expression for $(n;r) = \sum_{c=0}^{n-1} (n;\ r,c)$,

$r = 0, 1, \ldots, n-1$, as follows.

$$(n;r) = \sum_{c=0}^{n-1} (n;\ r,c) = \sum_{c=0}^{r} (n;\ r,c) + \sum_{c=r+1}^{n-1} (n;\ r,c)$$

$$= N\ (T(n;2r)) + \sum_{c=r+1}^{n-1} (n;\ r,c) \quad \text{by (17).}$$

Further,

$$(n;\ r,r+1) = \sum_{c=0}^{r+1} (n-1;\ r,c) \quad \text{by (V),}$$

$$= \sum_{c=0}^{r} (n-1;\ r,c) + (n-1;\ r,r+1)$$

$$= N(T(n-1;\ 2r)) + \sum_{c=0}^{r+1} (n-2;\ r,c) \text{ .}$$

Continuing this process till we get $\sum_{c=0}^{r+1} (r+1;\ r,c)$

$(= \sum_{c=0}^{r} (r+1;\ r,c)$ by (I)), $(n;\ r,r+1)$ becomes

$N(T(n-1;\ 2r)) + N(T(n-2;\ 2r)) + \ldots + N(T(r+1;\ 2r))$.

A similar simplification of other terms in $\sum_{c=r+1}^{n-1} (n;\ r,c)$ ultimately

leads to

$$(27) \qquad (n;r) = \sum_{\ell=0}^{n-r-1} \frac{n-r-\ell}{n-r+\ell} \binom{n-r+\ell}{\ell} N(T(n-\ell;\ 2r))$$

where $N(T(n-\ell;\ 2r))$ is given by either (20) or (24). It is interesting to observe that

$$(28) \qquad (n;0) = \sum_{\ell=0}^{n-1} \frac{n-\ell}{n+\ell} \binom{n+\ell}{\ell} = \frac{1}{n+1} \binom{2n}{n} ,$$

and

$$(29) \qquad (n;\ n-1) = N(T(n;\ 2n-2)) = \frac{1}{n-1} \binom{3n-3}{n-2} .$$

$\sum_{r=0}^{n-1} (n;r)$ denotes the total number of simple s.ps. of size n, which is equal to $\frac{1}{n}\binom{3n}{n-1}$. Therefore,

$$(30) \qquad \sum_{r=0}^{n-1} \sum_{\ell=0}^{n-r-1} \frac{n-r-\ell}{n-r+\ell} \binom{n-r+\ell}{\ell} N(T(n-\ell;\ 2r) = \frac{1}{n} \binom{3n}{n-1} .$$

Finally, we remark that numbers similar to $(n;\ r,c)$ can be constructed from the partition $\{T'(n;i)\}$ of S'_n. However, unlike $(n;\ r,c)$, these do not have a simple and elegant interpretation.

(iii) Consider a partition $\{U(n;\ i,j)\}$ of S_n which is the cross-partition of $\{S(n;i)\}$ and $\{T(n;j)\}$, defined in (i) and (ii). $U(n;\ i,j)$ denotes the set of vectors $\underset{\sim}{c}_n$ (defined in 2.5), whose last element $c_{n-1} = j$ and which contain exactly i

elements equal to zero (i.e. $c_1 = c_2 = \ldots = c_i = 0$; $c_k > 0$, $i < k \leq n-1$). We note that $U(n; i,0)$ $(i = 0, 1, \ldots, n-2)$ and $U(n; n-1,j)$ $(j = 1, 2, \ldots, 2n-2)$ are empty, whereas $U(n; n-1,0)$ contains only one vector, viz. $(0, 0, \ldots, 0)$. Without any difficulty, we can verify that the number of vectors in $U(n; i,j)$ is equal to the number of vectors in A_n which contain exactly $i+2$ zeros and whose $a_1 = [\frac{j+1}{2}]$ and $a_{n+1} \leq [\frac{j}{2}]$, the square bracket being used with its usual meaning. It is possible to express the $n \times n$ matrix (Table 3) M_n as

$$M_n = \sum_{i=0}^{n-1} M_{n,i} ,$$

where $M_{n,i}$ is the matrix constructed from $S(n;i)$ in the same way as M_n was from S_n. This break-up provides us some further identities.

3.3 A Coin Tossing Problem

Let us suppose that we are given two coins 1 and 2 with probabilities p_1 and p_2 of obtaining heads and consequently the probabilities q_1 and q_2 of obtaining tails, where $q_i = 1 - p_i$, $i = 1, 2$. We shall assume that $p_1 + p_2 > 1$. Consider a game H_N , played with the following rules:

1) Toss coin 1 and 2 alternately, making the first trial with coin 1.

2) Stop making further trials when the total number of

heads obtained with both coins exceeds the total number of tails obtained by exactly $N(N \geq 1)$ for the first time.

Following [*], we introduce the game H'_N which is played with the same rule as H_N except that the first trial is made with coin 2 instead of coin 1. Since we have assumed $p_1 + p_2 > 1$, the probability that both the games H_N and H'_N will terminate in a finite number of trials approaches unity as closely as we please. Let $H_N(s)$ and $H'_N(s)$ represent the generating functions for the probability distributions of H_N and H'_N respectively. We shall prove the following Theorem.

<u>Theorem 11</u> : The distributions H_N and H'_N are complete.

<u>Proof</u>: After the first trial in H_N, the game is H'_{N-1} or H'_{N+1} and therefore,

$$(31) \qquad H_N(s) = p_1 s\, H'_{N-1}(s) + q_1 s\, H'_{N+1}(s) .$$

Similarly,

$$(32) \qquad H'_N(s) = p_2 s\, H_{N-1}(s) + q_2 s\, H_{N+1}(s).$$

Set

$$(33) \qquad H_0(s) = H'_0(s) = 1 .$$

By symmetry,

(34) $$H_{2k}(s) = H'_{2k}(s) \quad , \quad k = 1, 2, \ldots \; .$$

We may easily check that

(35) $$H_2(s) = H_1(s) \; H'_1(s) \quad ,$$

and

(36) $$H'_2(s) = H'_1(s) \; H_1(s) \; .$$

Some elementary simplification gives us

(37) $$H_1(s) = \frac{1 + u - v - \sqrt{1 + u^2 + v^2 - 2u - 2v - 2uv}}{2q_2 s} \quad ,$$

and

(38) $$H'_1(s) = \frac{1 - u + v - \sqrt{1 + u^2 + v^2 - 2u - 2v - 2uv}}{2q_2 s} \quad ,$$

where

$$u = p_1 \, q_2 s^2 \quad \text{and} \quad v = q_1 \, p_2 s^2 \quad .$$

From the identity

(39) $$\sqrt{1+u^2+v^2-2u-2v-2uv} = 1-u-v-2 \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \frac{\binom{\ell+k-1}{\ell}\binom{\ell+k-1}{k}}{\ell+k-1} \, u^{\ell} \, v^{k} \; ,$$

which is proved in [*] or can be proved directly, (37) and (38) become

(40) $$H_1(s) = p_1 s + \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \frac{\binom{\ell+k-1}{\ell}\binom{\ell+k-1}{k}}{\ell+k-1} \, p_1^{\ell} \, q_1^{k} \, p_2^{k} \, q_2^{\ell-1} \, s^{2(\ell+k)-1}$$

and

$$(41)\qquad H_1'(s) = p_2 s + \sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty} \frac{\binom{\ell+k-1}{\ell}\binom{\ell+k-1}{k}}{\ell+k-1}\, p_1^{\ell}\, q_1^{k-1}\, p_2^{k}\, q_2^{\ell}\, s^{2(\ell+k)-1} .$$

Also, we obtain

$$(42)\qquad H_2(s) = H_2'(s) = \sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty} \frac{\binom{\ell+k-1}{\ell}\binom{\ell+k-1}{k}}{\ell+k-1}\, p_1^{\ell}\, q_1^{k-1}\, p_2^{k}\, q_2^{\ell-1}\, s^{2(\ell+k)-2} .$$

That the distributions H_1, H_1' and $H_2 (= H_2')$ are complete follow from [16]. We can express $H_N(s)$ and $H_N'(s)$ as a power series of the above form, and hence the theorem is proved.

It is worthwhile to mention here that results on completeness for a class of coin tossing problems have been discussed in [9].

Following the ideas and the results in Chapter I, we derive the distribution of H_N in two different ways and establish some combinatorial identities. The game H_N can only end at the $(2n+N)$th trial, $n = 0, 1, \dots$, and we shall have n tails and $n+N$ heads in the sequence of trials. Let $H_{N,n}$ be the set of sequences in H_N which contain exactly $2n+N$ trials. Representing a head or a tail by a unit horizontal or vertical step respectively, we notice that each sequence in $H_{N,n}$ corresponds to a minimal lattice path from $(0,0)$ to $(n+N, n)$ not touching the line $y = x-N$ except at $(n+N, n)$ and conversely. Let the set of lattice paths corresponding to $H_{N,n}$ be denoted by $L'(N,n)$. It is well-known that there exists a 1:1 correspondence between $L'(m-n, n)$, $m > n$, and the set $L(m,n;\,1)$ (as defined

in 1.3) of lattice paths from $(0,0)$ to (m,n) not touching the line $x = y$ except at the origin. So $H_{N,n}$ is isomorphic to $L(n+N,n; 1)$ and hence the number of sequences in $H_{N,n}$ is

$$(43) \qquad \frac{N}{2n+N} \binom{2n+N}{n} .$$

To determine the number of sequences in $H_{N,n}$ by another method, we need to define a <u>turn</u> in any lattice path and a <u>basic path</u> (B.P.). These definitions correspond to subsidiary and basic patterns in [14].

Either of the following two situations is called a turn in any lattice path:

Turn (1): A unit horizontal step with coin 1 followed by a unit vertical step with coin 2;

Turn (2): A unit vertical step with coin 1 followed by a unit horizontal step with coin 2.

A basic path (B.P.) is a lattice path with no turns. Hence in a B.P., any outcome of coin 2 is identical with the outcome of coin 1 at the previous trial. A horizontal (vertical) step with coin 1 followed by a horizontal (vertical) step with coin 2 is called a type 1(2) component. In addition, when N is odd, a type 3 component-viz. the last horizontal step with coin 1 - will occur. By the addition of turns to B.Ps., all paths in $L'(N,n)$ can be generated. For $N = 7$, we give an example to illustrate these definitions, in the figures below.

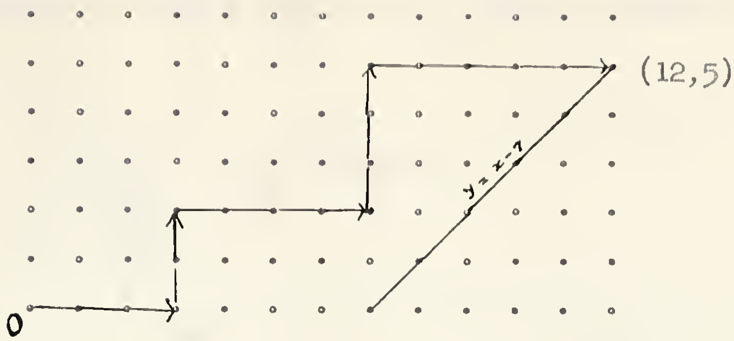


Figure 6

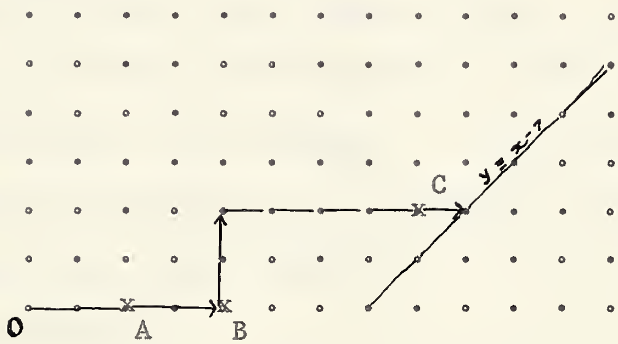
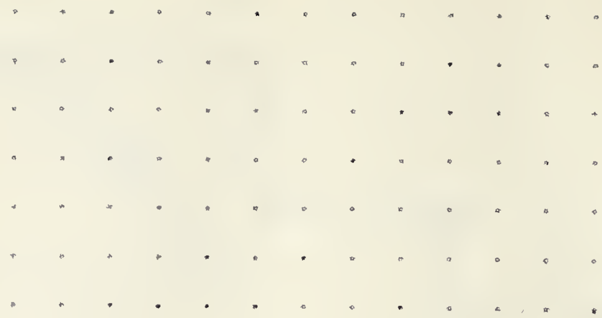


Figure 7

The path in $L'(7,5)$ of Figure 6 is generated from the B.P. shown in Figure 7, by inserting two turns (turn (1) and turn (2) successively) at A before the second component and one turn (1) at B before the third component of the B.P.

Let the set of B.Ps. of $L'(N,n)$ which can generate all paths in $L'(N,n)$ be denoted by $L^*(N,n)$. From the definition of B.P., it follows that the set $L^*(N,n)$ can be partitioned into $\{L^*(N,n,r)\}$ $r = 0, 1, \ldots, [\frac{n}{2}]$, where $L^*(N,n,r)$ is the set of B.Ps. from $(0,0)$ to $(2r+N, 2r)$ not touching $y = x-N$ except at $(2r+N, 2r)$. $L^*(N,n,r)$ is obviously isomorphic to $L'([\frac{N+1}{2}], r)$



and therefore to $L(r + [\frac{N+1}{2}], r;1)$. Thus the number of B.Ps. in $L^*(N,n,r)$ is

$$(44) \qquad \frac{[\frac{N+1}{2}]}{2r + [\frac{N+1}{2}]} \binom{2r + [\frac{N+1}{2}]}{r} \quad .$$

Now we remark that if N is even, turns of both kinds can be inserted before every component, whereas if N is odd, no turn (1) can be inserted before the component of type 3 and those components of type 2 which touch the line $y = x - N - 1$. Positions where a turn cannot be inserted will be called unfavourable positions (U.Ps.). For instance, C is an U.P. in Figure 7. This leads us to treat the set $L'(2k,n)$, $k = 1, 2, \ldots$ of even paths and the set $L'(2k+1, n)$, $k = 0, 1, 2, \ldots$ of odd paths separately.

For structural simplicity, we first consider even paths. It is easily seen that a B.P. in $L^*(2k,n,r)$ yields

$$(45) \qquad \binom{n + k - 1}{n - 2r} 2^{n-2r}$$

paths in $L'(2k,n)$. (44) and (45) give an alternative expression for the number of paths in $L'(2k,n)$ i.e. for the number of sequences in $H_{2k,n}$ given by (43). Thus we have the identity

$$(46) \qquad \sum_{r=0}^{[\frac{n}{2}]} \frac{k}{2r+k} \binom{2r+k}{r}\binom{n+k-1}{n-2r} 2^{n-2r} = \frac{k}{n+k} \binom{2n+2k}{n} \quad .$$

Denoting by $[2k,n,r]_t$, the number of paths in $L'(2k,n)$ which are generated from $L^*(2k,n,r)$ and have t units

of vertical step with coin 1, we obtain by a simple calculation,

$$(47) \qquad [2k,\, n,\, r]_t = \frac{k}{n+k} \binom{n+k-t}{n-t} \binom{n+k}{t} \binom{t}{r} .$$

Thus the probability that the game H_{2k} stops at the $(2n+2k)$th trial is

$$(48) \qquad \sum_{r=0}^{[\frac{n}{2}]} \sum_{t=r}^{n-r} \frac{k}{n+k} \binom{n+k-t}{n-t} \binom{n+k}{t} \binom{t}{r} p_1^{n+k-t} q_1^{t} p_2^{t+k} q_2^{n-t}$$

and hence

$$(49) \qquad (p_1 p_2)^k \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{2}]} \sum_{t=r}^{n-r} \frac{k}{n+k} \binom{n+k-t}{n-t} \binom{n+k}{t} \binom{t}{r} (p_1 q_2)^{n-t} (q_1 p_2)^{t} = 1 .$$

As indicated earlier, the solution for H_{2k+1} $(k = 0, 1, \ldots)$ is not immediate, due to the existence of the U.Ps. in odd paths. For convenience in calculation, let $\{L^*(2k+1, n, r, b)\}$ $b = 1, 2, \ldots, r+1$ be a partition of $L^*(2k+1, n, r)$, where $L^*(2k+1, n, r, b)$ is the subset of B.Ps. in $L^*(2k+1, n, r)$, having exactly b U.Ps. But $L^*(2k+1, n, r, b)$ is isomorphic to the subset of paths in $L'(k,r)$ which touch the line $y = x-k$ exactly b times, and therefore, to the subset of paths in $L(r+k, r;1)$ which touch the line $y = x$ exactly b times. By the application of the result in section 1.3 and the method of induction, we obtain the number of B.Ps. in $L^*(2k+1, n, r, b)$ as

$$(50) \qquad \frac{k+b-1}{2r+k-b+1} \binom{2r+k-b+1}{r+k} ,$$

for $k, r = 0, 1, \ldots$, and $b = 1, 2, \ldots, r+1$, except for k and r together equal to 0. The number of B.Ps. in $L^*(1, n, 0, b)$ is equal to 1, if $b = 1$, and equal to 0, otherwise.

We shall get identities for the game H_{2k+1}, similar to (46) and (49) as follows. In a B.P. of $L^*(2k+1, n, r, b)$, we first insert $n-2r-\ell$ turns of type (1) in the favourable positions (the number of which is $2r+k-b+1$). Then we insert ℓ turns of type (2) for which the positions available are $n+k-\ell+1$. Therefore,

$$(51)\qquad \sum_{r=0}^{[\frac{n}{2}]} \sum_{b=1}^{r+1} \sum_{\ell=0}^{n-2r} \frac{k+b-1}{2r+k-b+1} \binom{2r+k-b+1}{r+k} \binom{n+k-b-\ell}{n-2r-\ell} \binom{n+k}{\ell} = \frac{2k+1}{2n+2k+1} \binom{2n+2k+1}{n} .$$

Defining $[2k+1, n, r]_t$, as before, the identity in probability is given below.

$$(52)\qquad p_1(p_1\, p_2)^k \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{2}]} \sum_{t=r}^{n-r} \sum_{b=1}^{r+1} \frac{k+b-1}{2r+k-b+1} \binom{2r+k-b+1}{r+k} \binom{n+k+r-t-b}{n-r-t} \binom{n+k}{t-r}$$

$$(q_1\, p_2)^t\, (p_1\, q_2)^{n-t} = 1 .$$

The relations (51) and (52) are true for all k and r such that k and r together are not equal to zero. When $k = 0$, we have

$$(53)\qquad p_1 \left\{ \sum_{n=0}^{\infty} (q_1\, p_2)^n + \sum_{n=2}^{\infty} \sum_{r=1}^{[\frac{n}{2}]} \sum_{t=r}^{n-r} \sum_{b=2}^{r+1} \frac{b-1}{2r-b+1} \binom{2r-b+1}{r} \binom{n+r-t-b}{n-r-t} \binom{n}{t-r} \right.$$

$$\left. (q_1\, p_2)^t\, (p_1\, q_2)^{n-t} \right\} = 1 .$$

We conclude this section with an elegant interpretation of the game H_{2k} which enables us to calculate the duration of the game and the variance easily. The game H_{2k} can be considered as a random walk in one dimension with $2k$ as the absorbing barrier where the particle can move two units to the right with probability $p = p_1 p_2$, or two units to the left with probability $q = q_1 q_2$ or stay in its position with probability $r = p_1 q_2 + q_1 p_2$ $(p + q + r = 1)$. This type of random walk has been thoroughly discussed in [4] as a classical ruin problem. It can also be treated as an extension of the game g_k [10]. Using results in [10], the generating function of the probability that the game ends exactly at the $(2n+2k)$th trial can be written as

$$(54) \qquad \sum_{n=0}^{\infty} \frac{k}{2n+k} \binom{2n+k}{n} p^{n+k} q^{n} \left(\frac{s^2}{1-rs^2}\right)^{2n+k} .$$

The mean and variance of the number of trials to end the game is found to be

$$(55) \qquad \frac{2k}{p-q} \quad \text{and} \quad \frac{4k[p+q - (p-q)^2]}{(p-q)^3}$$

respectively.

The game H_{2k+1} does not permit an interpretation so simple as that of the game H_{2k}.

3.4 Some Number Theoretic Results

We quote, without proof, two results pertaining to sampling plans and their vectorial representations.

(i) Let $S(n;i)$, as defined in 3.2(i) be partitioned into $\{S(n;\, x,y)\}$ where any vector $\underline{c}(n;\, x,y) \in S(n;\, x,y)$ contains $n-i$ positive elements of which x are odd and y are even (i.e. $x + y = n - i$). Then the number $N(S(n;\, x,y))$ of vectors in $S(n;\, x,y)$ is equal to

$$(56) \qquad \frac{\binom{n+x}{x}\binom{n+y}{y}}{(n+x)\,(n+y)} \;\{ n^2 - n(x+y)\} \,.$$

Obviously,

$$(57) \qquad N(S(n;\, x,y)) \;=\; N(S(n;\, y,x)) \,.$$

$S(n;\, x,y)$ maps into the set $A(n;\, x,y) \subset A_n$, where any vector $\underline{a}(n;\, x,y) \in A(n;\, x,y)$ has x and y positive elements respectively to the left and right of the zeros. (56) and (57) imply that

$$(58) \qquad N(A(n;\, x,y)) = N(A(n;\, y,x)) = \frac{\binom{n+x}{x}\binom{n+y}{y}}{(n+x)(n+y)} \;\{ n^2 - n(x+y)\} \,.$$

Of course, the first part of (58) follows directly from the fact that a s.p. whose vector is $\underline{a}(n;\, x,y)$ when reflected about the line $x = y$, yields a s.p. whose vector is $\underline{a}(n;\, y,x)$.

A corollary from (i) is: consider the set $(a_1, \ldots, a_n)$ of vectors of positive integers satisfying

$$1 \leq a_1 \leq \ldots \leq a_n ,$$

and

$$a_i \leq 2i \qquad (i = 1, 2, \ldots, n).$$

Choose one of these vectors at random and then choose one of its elements at random. The probability is $\frac{1}{2}$ that the chosen element is congruent to 0 or 1 (mod 2).

(ii) The other result is as follows:

The set $S(n; x,y)$ is the (disjoint) union of the sets $S(n-1; i,j)$ $(i = 0, 1, \ldots x ; \ j = 0, 1, \ldots, y)$. Trivially $S(n; x,y)$ is empty if $x+y \geq n$ and $S(n; 0,0)$ contains one vector i.e. $(0, 0, \ldots, 0)$. This helps us to build $N(S(n; x,y))$ recursively.

The above results follow either from an extension of the 1:1 correspondence theorem or as a special case of more general results to be published shortly (Narayana: "An analogue of the multinomial theorem"). The interpretation of these results from the lattice theoretic point of view (either as a lattice of lattice paths or of non-decreasing and non-negative vectors) is straightforward; the details follow from our previous discussion.

BIBLIOGRAPHY

[1] Brainerd, B. and Narayana, T. V. "A Characterization of Simple Binomial Sampling Plans", (To be published)

[2] DeGroot, M. H. "Unbiased Sequential Estimation for Binomial Populations", <u>The Annals of Mathematical Statistics</u>, XXX (1959), pp. 80.

[3] Dvoretzky, A. and Motzkin, TH. "A Problem of Arrangements", <u>Duke Mathematical Journal</u>, XIV (1947), pp. 305.

[4] Feller, W. An Introduction to Probability Theory and its Applications, New York, John Wiley and Sons, Inc., 1957.

[5] Girshick, M. A., Mosteller, F. and Savage L. J. "Unbiased Estimates for Certain Binomial Sampling Plans with Applications", <u>The Annals of Mathematical Statistics</u>, XVII (1946), pp.13.

[6] Grossman, H. D. "Fun with Lattice Points", <u>Scripta Mathematica</u>, XII (1946), pp. 223.

[7] Grossman, H. D. "Fun with Lattice Points", <u>Scripta Mathematica</u>, XVI (1950), pp. 119.

[8] Lehman, E. L. and Stein, C. "Completeness in the Sequential Case", <u>The Annals of Mathematical Statistics</u>, XXI (1950), pp. 376.

[9] Narayana, T. V., Chorneyko, I. and Sathe, Y. S. "Sufficient Partitions for a Class of Coin Tossing Problems", <u>Biometrische Zeitschrift</u>, II (1960), pp. 269.

[10] Narayana, T. V. "A Problem in the Theory of Probability", <u>Journal of the Indian Society of Agricultural Statistics</u>, VI (1954).

[11] Narayana, T. V. "A Partial Order and its Applications to Probability Theory", <u>Sankhyā</u> : <u>The Indian Journal of Statistics</u>, XXI (1959), pp. 91.

[12] Narayana, T. V. "Sur les Treillis Formés par les Partitions d'un Entier et Leurs Applications à la Théorie des Probabilites", <u>Comptes Rendus des Séances de l'Académie des Sciences</u>, t 240 (1955), pp. 1188.

[13] Narayana, T. V. and Fulton, G. E. "A Note on the Compositions of an Integer", <u>Canadian Mathematical Bulletin</u>, I (1958), pp. 169.

[14] Narayana, T. V., Mohanty, S. G. and Ladouceur, J. C. "A Combinatorial Problem and its Applications to Probability Theory - II", (to be published)

[15] Poincare, H. Calcul des Probabilités, Paris Gauthier Villars et Cie, 1913.

[16] Sathe, Y. S. "Use of Series Expansion in Estimation Problems for Distributions Involving More Than One Parameter", The Annals of Mathematical Statistics, XXX (1959), pp. 613.

[17] Gould, H. W. "Combinatorial Identities, A Standardized Set of Tables Listing 500 Binomial Coefficient Summations", National Security Agency, Fort George G. Meade, Maryland 1959.